



ELSEVIER

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt


The Riemann zeta in terms of the dilogarithm

 Sergio Albeverio^{a,b,c,d,e,f,g}, Claudio Cacciapuoti^{h,b,*}
^a SFB611, Bonn, Germany

^b Hausdorff Center for Mathematics, Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

^c BIBOS, Bielefeld–Bonn, Germany

^d IZKS, Bonn, Germany

^e CERFIM, Locarno, Switzerland

^f Acc. Arch., USI, Mendrisio, Switzerland

^g Dip. Matematica, Università di Trento, Italy

^h Graduate School of Science, Mathematical Institute, Tohoku University, 6-3 Aoba, Aramaki, Aoba, Sendai, Miyagi 980-8578, Japan

ARTICLE INFO

Article history:

Received 26 January 2011

Revised 26 May 2012

Accepted 4 June 2012

Available online 18 September 2012

Communicated by David Goss

MSC:

primary 11M26, 11M06, 11F27

secondary 42A38

Keywords:

Classical Riemann's zeta function

Dilogarithm function

Mellin transform

 Meromorphic character of ζ

Zero-free regions of the zeta function

Bounds on the zeta function

Integral representations of the zeta function

Series of zeta functions

Series of gamma functions

Generalized symmetrized Müntz formula

ABSTRACT

We give a representation of the classical Riemann ζ -function in the half plane $\text{Re } s > 0$ in terms of a Mellin transform involving the real part of the dilogarithm function with an argument on the unit circle (associated Clausen Gl_2 -function). We also derive corresponding representations involving the derivatives of the Gl_2 -function. A generalized symmetrized Müntz-type formula is also derived. For a special choice of test functions it connects to our integral representation of the ζ -function, providing also a computation of a concrete Mellin transform. Certain formulae involving series of zeta functions and gamma functions are also derived.

© 2012 Elsevier Inc. All rights reserved.

* Corresponding author at: Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany.

E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), cacciapuoti@him.uni-bonn.de (C. Cacciapuoti).

1. Introduction

The classical Riemann zeta function ζ is one of the most intriguing and central objects of mathematics. Both the location and distribution of its zeros have deep connections with the distribution of prime numbers. The set of zeros of ζ consists exclusively of the strictly negative even integers (“trivial zeros”) and, according to the as of yet still unproven Riemann’s hypothesis [38], of a countable number of points on the critical line $\operatorname{Re} s = \frac{1}{2}$ in the complex s -plane (that there are indeed infinitely many zeros on the critical line is a classical result of Hardy, see, e.g., [17]). There are many connections of Riemann’s zeta function with areas of mathematics and its applications, as well as with other conjectures, see [38] for the original work by B. Riemann (and also, e.g., [5,17] for comments resp. further historical comments) and [23,25,36,44] for basic specific books on the Riemann’s zeta function, as well as the survey papers [4,15]. For connections with other problems in analytic number theory see, e.g., [6,24], for relations with new developments in random matrix theory and other areas of mathematics see, e.g., [13–15,26,39], for numerical results and other relations to methods inspired by physics see [3], and references therein. Some results establishing “zero-free regions” in $\{\operatorname{Re} s \neq \frac{1}{2}\} \setminus \{-2\mathbb{N}\}$ are known, see, e.g., [1,19,24,25,44,45], and references therein. These, in turn, are related with estimates on the remainder in the classical prime number theorem see, e.g., [24,25].

The dilogarithm function $Li_2(z)$, $z \in \mathbb{C}$, was introduced by Euler in 1768 by its series representation $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ (for $|z| \leq 1$) and further studied, e.g. by Hill in 1828 see, e.g., [28,29]. It is a member of a family of polylogarithm functions which have many connections with other areas of mathematics, like number theory, differential and algebraic geometry, but also with physics (quantum electrodynamics, scattering theory), and engineering (signal analysis, electrical network theory, network problems, waveguide theory). An important function associated with Li_2 is Clausen’s Gl_2 -function defined by

$$\operatorname{Re} Li_2(e^{i\theta}) = Gl_2(\theta) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}.$$

It is given in $[0, 2\pi]$ by $\frac{\pi^2}{6} - \frac{\theta(2\pi-\theta)}{4}$. In the present paper we relate $Gl_2(\theta)$, which we call for simplicity $p(\theta)$, to the ζ -function, via a Mellin transform. More precisely, we prove a representation of $\zeta(s)$ for $\operatorname{Re} s > 0$ in terms of $p(\theta)$ of the form

$$\zeta(s) = \frac{2s(1+s)}{(2\pi)^{1-s}} \left\{ \frac{\pi^2}{6} \frac{1}{1+s} - \frac{\pi}{2} \frac{1}{s} - \frac{1}{4} \frac{1}{1-s} - D(-2-s) \right\},$$

where $D(\alpha)$ is defined for $\operatorname{Re} \alpha < -2$ as $D(\alpha) \equiv \int_1^{+\infty} y^\alpha p(y) dy$ (see Theorem 3.1) and is the Mellin transform at $\alpha + 1$ of the function $p_0(y) = \chi_{[1, \infty)}(y)p(y)$ (χ_A denoting the characteristic function of the set A). This provides a, to the best of our knowledge, new integral representation of $\zeta(s)$ for $\operatorname{Re} s > 0$. Furthermore we provide a series representation for the function $D(\alpha)$ (see Remark 2.4). We note that our representation of the ζ -function differs in several ways from other known representations, like those given in [4,17,38–40] (see also [43, Sect. 3]). As corollaries we obtain a new proof that ζ is a meromorphic function in $\operatorname{Re} s > 0$, provide new proofs of certain results on zero-free regions for the ζ -function and bounds of it inside the critical strip. We also derive other integral representations of ζ in terms of derivatives of the function p (Section 4).

Our integral representation of the ζ -function also yields an explicit formula for the Mellin transform of the Fourier transform of the test function $\varphi(x) = (1 - |x|)\chi_{[-1,1]}(x)$ (see Remark 5.11). We also get an explicit formula for the Mellin transform of the function $\frac{1 - \cos(2\pi x)}{2\pi^2 x^2} \chi_{[0,1]}(x)$ in terms of Γ -functions. The summation of certain series involving factorial factors (see Section 5) resp. the zeta functions at equally spaced arguments is also performed as an application of our integral representation (see Appendix A).

The structure of the present paper is as follows. In Section 2 after the introduction of $p(\theta) = Gl_2(\theta)$ we prove a lemma giving a representation of $D_N(\alpha) \equiv \int_1^{2\pi N} y^\alpha p(y) dy$ for $\operatorname{Re} \alpha < -1$, $N \in \mathbb{N}$

(incomplete Mellin transform of p). We then prove that $\lim_{N \rightarrow \infty} D_N(\alpha) \equiv D(\alpha)$ exists and can be expressed by $\zeta(-2 - \alpha)$ for $\operatorname{Re} \alpha < -2$. Then expressions for $D(\alpha)$ as series involving the gamma function are given. In Section 3 the representation of ζ in terms of D is used in particular to derive in a simple way a zero-free region for ζ , close to the real line (cf. Remark 3.4). In Section 4 we derive two other representations of ζ in terms of integrals involving, instead of p , its distributional first derivative (Proposition 4.2) resp. a piecewise constant function (Proposition 4.6). In particular they lead to upper bounds on $|\zeta(s)|$, for $\operatorname{Re} s > 0$. For a comparison with other bounds obtained essentially by trigonometric sums methods see Remark 4.4 below.

In Section 5 we derive a generalized Müntz formula for ζ , relative to general “test functions” which are such that they as well as their Fourier transforms are in $L^1(\mathbb{R})$ (for more restrictive choices of f the formula was originally proven in [32], see also [7], and Remark 5.2 below). We also prove, exploiting a Poisson summation formula, a symmetrized version of our generalized Müntz formula, somewhat related to the one discussed in [1], and which might be of interest in itself. For the special choice $f = \varphi$ (with φ as above) these formulae yield an explicit computation of the Mellin transform of the Fourier transform of f in terms of ζ .

In Appendix A we provide another derivation for the basic function D of Section 2. This derivation involves the computation of certain series $\tilde{A}(\alpha)$ (simply related to $D(\alpha)$) in terms of series involving the ζ -functions taken at equally spaced arguments (cf. Lemma A.2). The latter in turn are expressed in Corollary A.4 in simple terms and a zeta function at a single point. These relations might have an interest in themselves (in any case we were not able to locate them in the extensive survey on series involving the zeta function presented in [40]).

2. An integral involving the associated Clausen Gl_2 -function

Let p be the 2π -periodic real-valued function on \mathbb{R} given for $\theta \in [0, 2\pi]$ by

$$p(\theta) = \frac{\pi^2}{6} - \frac{\theta(2\pi - \theta)}{4}. \quad (2.1)$$

The function $p(\theta)$ is denoted by $Gl_2(\theta)$ in [28, p. 181] and is called the associated Clausen function (of order 2). It is also given by

$$p(\theta) = \sum_{n=1}^{+\infty} \frac{\cos(n\theta)}{n^2}, \quad \theta \in \mathbb{R}, \quad (2.2)$$

see, e.g., [28, p. 242]. One has

$$p(\theta) = \pi^2 B_2\left(\frac{\theta}{2\pi}\right), \quad \theta \in [0, 2\pi], \quad (2.3)$$

where $B_2(x)$ is the second Bernoulli polynomial, defined as the coefficient of $t^2/2!$ in the expansion of $te^{tx}/(e^t - 1)$ in powers of t , i.e.

$$B_2(x) = x^2 - x + \frac{1}{6} \quad (2.4)$$

(cf., e.g., [18, p. 361], and [28, p. 186]).

p coincides with the real part of Euler's (1768) dilogarithm function $Li_2(\cdot)$ with an argument on the unit circle, i.e.

$$p(\theta) = \operatorname{Re} Li_2(e^{i\theta}), \quad \theta \in [0, 2\pi], \quad (2.5)$$

$Li_2(z)$ being defined for $|z| \leq 1$ by

$$Li_2(z) \equiv \sum_{n=1}^{+\infty} \frac{z^n}{n^2} \quad (2.6)$$

(see, e.g., [28,29] and [42, p. 106]).

p is bounded continuous on \mathbb{R} (with $\max p(\theta) = \max |p(\theta)| = \pi^2/6$, $\min p(\theta) = -\pi^2/12$). All its derivatives exist and are continuous except for the points $2\pi k$, $k \in \mathbb{Z}$ (where they have to be defined, e.g., in the distributional sense). The function $y \rightarrow y^\alpha p(y)$ is in $L^1([1, +\infty))$ (with respect to Lebesgue's measure on $[1, +\infty)$), for any $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha < -1$ (since p is bounded).

One has the integral representation

$$p(x) = \operatorname{Re} Li_2(e^{ix}) = -\frac{1}{2} \int_0^1 \frac{1}{y} \log(1 - 2y \cos x + y^2) dy,$$

see, e.g., [28, p. 106].

We shall study the function

$$D(\alpha) \equiv \int_1^{+\infty} y^\alpha p(y) dy, \quad (2.7)$$

relating it, in Section 3, to the classical Riemann zeta function ζ at $(-2-\alpha)$, $\operatorname{Re} \alpha < -2$ (i.e. $\zeta(s)$ with $\operatorname{Re} s > 0$).

Lemma 2.1. For any integer $N \in \mathbb{N}$ and $\operatorname{Re} \alpha < -1$, let $D_N(\alpha)$ be defined by

$$D_N(\alpha) \equiv \int_1^{2\pi N} y^\alpha p(y) dy, \quad (2.8)$$

then

$$\begin{aligned} D_N(\alpha) = & -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} \zeta_N(-2-\alpha) \\ & + \frac{\pi^2}{6} \frac{(2\pi N)^{\alpha+1}}{\alpha+1} + \frac{\pi}{2} \frac{(2\pi N)^{\alpha+2}}{(\alpha+2)(\alpha+1)} \\ & + \frac{1}{2} \frac{(2\pi N)^{\alpha+3}}{(\alpha+3)(\alpha+2)(\alpha+1)}, \end{aligned} \quad (2.9)$$

with

$$\zeta_N(s) \equiv \sum_{n=1}^N \frac{1}{n^s}. \quad (2.10)$$

Proof. Splitting the integration domain $[1, 2N\pi)$ in (2.8) into

$$[1, 2\pi) \cup [2\pi, 2N\pi) = [1, 2\pi) \cup \bigcup_{k=1}^{N-1} [2\pi k, 2\pi(k+1))$$

we get

$$\int_1^{2N\pi} y^\alpha p(y) dy = \int_1^{2\pi} y^\alpha p(y) dy + \sum_{k=1}^{N-1} \int_{2\pi k}^{2\pi(k+1)} y^\alpha p(y) dy. \quad (2.11)$$

Set

$$I^\alpha \equiv \int_1^{2\pi} y^\alpha p(y) dy. \quad (2.12)$$

Then, by the definition (2.1) of p in $[0, 2\pi]$:

$$\begin{aligned} I^\alpha &= \int_1^{2\pi} y^\alpha \left(\frac{\pi^2}{6} - \frac{\pi}{2}y + \frac{1}{4}y^2 \right) dy = \frac{\pi^2}{6} \frac{y^{\alpha+1}}{\alpha+1} \Big|_1^{2\pi} - \frac{\pi}{2} \frac{y^{\alpha+2}}{\alpha+2} \Big|_1^{2\pi} + \frac{1}{4} \frac{y^{\alpha+3}}{\alpha+3} \Big|_1^{2\pi} \\ &= \frac{\pi^2}{6} \frac{1}{\alpha+1} ((2\pi)^{\alpha+1} - 1) - \frac{\pi}{2} \frac{1}{\alpha+2} ((2\pi)^{\alpha+2} - 1) + \frac{1}{4} \frac{1}{\alpha+3} ((2\pi)^{\alpha+3} - 1). \end{aligned} \quad (2.13)$$

Set

$$II_N^\alpha \equiv \sum_{k=1}^{N-1} \int_{2\pi k}^{2\pi(k+1)} y^\alpha p(y) dy, \quad (2.14)$$

so that by (2.11), (2.12) and (2.14):

$$D_N(\alpha) = \int_1^{2\pi N} y^\alpha p(y) dy = I^\alpha + II_N^\alpha. \quad (2.15)$$

Then, by the definition (2.1) of p in $[2\pi k, 2\pi(k+1))$:

$$\begin{aligned} \int_{2\pi k}^{2\pi(k+1)} y^\alpha p(y) dy &= \int_{2\pi k}^{2\pi(k+1)} y^\alpha \left[\frac{\pi^2}{6} - \frac{\pi}{2}(y - 2\pi k) + \frac{1}{4}(y - 2\pi k)^2 \right] dy \\ &= \int_{2\pi k}^{2\pi(k+1)} \left[\frac{\pi^2}{6} + \frac{\pi}{2}(2\pi k) + \frac{1}{4}(2\pi k)^2 \right] y^\alpha - \left[\frac{\pi}{2} + \frac{1}{2}(2\pi k) \right] y^{\alpha+1} + \frac{1}{4} y^{\alpha+2} dy \\ &= \left[\frac{\pi^2}{6} + \frac{\pi}{2}(2\pi k) + \frac{1}{4}(2\pi k)^2 \right] \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \end{aligned}$$

$$\begin{aligned}
& - \left[\frac{\pi}{2} + \frac{1}{2}(2\pi k) \right] \frac{1}{\alpha+2} [(2\pi(k+1))^{\alpha+2} - (2\pi k)^{\alpha+2}] \\
& + \frac{1}{4} \frac{1}{\alpha+3} [(2\pi(k+1))^{\alpha+3} - (2\pi k)^{\alpha+3}].
\end{aligned}$$

By the definition (2.14) of II_N^α and rearranging the terms in the latter equality we then get

$$\begin{aligned}
II_N^\alpha = & \sum_{k=1}^{N-1} \left[\frac{\pi^2}{6} \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \right. \\
& - \frac{\pi}{2} \frac{1}{\alpha+2} [(2\pi(k+1))^{\alpha+2} - (2\pi k)^{\alpha+2}] \\
& + \frac{1}{4} \frac{1}{\alpha+3} [(2\pi(k+1))^{\alpha+3} - (2\pi k)^{\alpha+3}] \\
& + \frac{\pi}{2} (2\pi k) \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \\
& - \frac{1}{2} (2\pi k) \frac{1}{\alpha+2} [(2\pi(k+1))^{\alpha+2} - (2\pi k)^{\alpha+2}] \\
& \left. + \frac{1}{4} (2\pi k)^2 \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \right].
\end{aligned}$$

We use the equalities

$$\sum_{k=1}^{N-1} [(k+1)^{\alpha+j} - k^{\alpha+j}] = \sum_{k=1}^{N-1} (k+1)^{\alpha+j} - \sum_{k=1}^{N-1} k^{\alpha+j} = N^{\alpha+j} - 1, \quad j = 1, 2, 3,$$

and obtain

$$\begin{aligned}
II_N^\alpha = & \frac{\pi^2}{6} \frac{(2\pi)^{\alpha+1}}{\alpha+1} [N^{\alpha+1} - 1] - \frac{\pi}{2} \frac{(2\pi)^{\alpha+2}}{\alpha+2} [N^{\alpha+2} - 1] + \frac{1}{4} \frac{(2\pi)^{\alpha+3}}{\alpha+3} [N^{\alpha+3} - 1] \\
& + \sum_{k=1}^{N-1} \left[\frac{\pi}{2} (2\pi k) \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \right. \\
& - \frac{1}{2} (2\pi k) \frac{1}{\alpha+2} [(2\pi(k+1))^{\alpha+2} - (2\pi k)^{\alpha+2}] \\
& \left. + \frac{1}{4} (2\pi k)^2 \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \right]. \tag{2.16}
\end{aligned}$$

We notice that

$$\begin{aligned}
& \sum_{k=1}^{N-1} k [(k+1)^{\alpha+j} - k^{\alpha+j}] \\
& = \sum_{k=1}^{N-1} k(k+1)^{\alpha+j} - \sum_{k=1}^{N-1} k k^{\alpha+j}
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{k=1}^{N-1} (k+1)^{\alpha+j} + \sum_{k=1}^{N-1} (k+1)^{\alpha+j+1} - \sum_{k=1}^{N-1} k^{\alpha+j+1} \\
&= 1 - \zeta_N(-\alpha-j) + N^{\alpha+j+1} - 1 = -\zeta_N(-\alpha-j) + N^{\alpha+j+1}, \quad j=1, 2 \quad (2.17)
\end{aligned}$$

where $\zeta_N(-\alpha-j)$ is the function defined in (2.10), and we used the equality $\sum_{k=1}^{N-1} (k+1)^{\alpha+j} = \sum_{m=2}^N m^{\alpha+j} = -1 + \zeta_N(-\alpha-j)$. We also notice the equality

$$\begin{aligned}
\sum_{k=1}^{N-1} k^2 [(k+1)^{\alpha+1} - k^{\alpha+1}] &= \sum_{k=1}^{N-1} (k+1)^{\alpha+1} - 2 \sum_{k=1}^{N-1} (k+1)^{\alpha+2} + \sum_{k=1}^{N-1} (k+1)^{\alpha+3} - \sum_{k=1}^{N-1} k^{\alpha+3} \\
&= -1 + \zeta_N(-\alpha-1) + 2 - 2\zeta_N(-\alpha-2) + N^{\alpha+3} - 1 \\
&= \zeta_N(-\alpha-1) - 2\zeta_N(-\alpha-2) + N^{\alpha+3}. \quad (2.18)
\end{aligned}$$

Using equalities (2.17) and (2.18) in (2.16) we have

$$\begin{aligned}
I_N^\alpha &= \frac{\pi^2}{6} \frac{(2\pi)^{\alpha+1}}{\alpha+1} [N^{\alpha+1} - 1] - \frac{\pi}{2} \frac{(2\pi)^{\alpha+2}}{\alpha+2} [N^{\alpha+2} - 1] + \frac{1}{4} \frac{(2\pi)^{\alpha+3}}{\alpha+3} [N^{\alpha+3} - 1] \\
&\quad + \frac{\pi}{2} \frac{(2\pi)^{\alpha+2}}{\alpha+1} [-\zeta_N(-\alpha-1) + N^{\alpha+2}] - \frac{1}{2} \frac{(2\pi)^{\alpha+3}}{\alpha+2} [-\zeta_N(-\alpha-2) + N^{\alpha+3}] \\
&\quad + \frac{1}{4} \frac{(2\pi)^{\alpha+3}}{\alpha+1} [\zeta_N(-\alpha-1) - 2\zeta_N(-\alpha-2) + N^{\alpha+3}] \\
&= -\frac{\pi^2}{6} \frac{(2\pi)^{\alpha+1}}{\alpha+1} + \frac{\pi}{2} \frac{(2\pi)^{\alpha+2}}{\alpha+2} - \frac{1}{4} \frac{(2\pi)^{\alpha+3}}{\alpha+3} + \frac{(2\pi)^{\alpha+3}}{2} \left[\frac{1}{\alpha+2} - \frac{1}{\alpha+1} \right] \zeta_N(-\alpha-2) \\
&\quad + \frac{\pi^2}{6} \frac{(2\pi)^{\alpha+1}}{\alpha+1} N^{\alpha+1} + (2\pi)^{\alpha+2} \frac{\pi}{2} \left[\frac{1}{\alpha+1} - \frac{1}{\alpha+2} \right] N^{\alpha+2} \\
&\quad + (2\pi)^{\alpha+3} \frac{1}{2} \left[\frac{1}{2(\alpha+3)} - \frac{1}{\alpha+2} + \frac{1}{2(\alpha+1)} \right] N^{\alpha+3}. \quad (2.19)
\end{aligned}$$

Using the latter equality and the formula (2.13) for I^α in (2.15) we get

$$\begin{aligned}
D_N(\alpha) &= -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} \\
&\quad + \frac{(2\pi)^{\alpha+3}}{2} \left[\frac{1}{\alpha+2} - \frac{1}{\alpha+1} \right] \zeta_N(-\alpha-2) \\
&\quad + \frac{\pi^2}{6} \frac{(2\pi)^{\alpha+1}}{\alpha+1} N^{\alpha+1} + (2\pi)^{\alpha+2} \frac{\pi}{2} \left[\frac{1}{\alpha+1} - \frac{1}{\alpha+2} \right] N^{\alpha+2} \\
&\quad + (2\pi)^{\alpha+3} \frac{1}{2} \left[\frac{1}{2(\alpha+3)} - \frac{1}{\alpha+2} + \frac{1}{2(\alpha+1)} \right] N^{\alpha+3},
\end{aligned}$$

which is the equality (2.9). \square

We observe that (2.9) can be written as

$$D_N(\alpha) = -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} A_N(-2-\alpha) + \frac{\pi^2}{6} \frac{(2\pi N)^{\alpha+1}}{\alpha+1}$$

with

$$\begin{aligned} A_N(s) &\equiv \zeta_N(s) - \frac{1}{2} \frac{1}{N^s} - \frac{1}{1-s} \frac{1}{N^{s-1}} \\ &= \sum_{n=1}^N \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - \frac{N^{-s}}{2}. \end{aligned} \quad (2.20)$$

This will be used in the proof of the following:

Lemma 2.2. *Let $D_N(\alpha)$ be defined as in Lemma 2.1, then for $\operatorname{Re} \alpha < -2$:*

$$\lim_{N \rightarrow +\infty} D_N(\alpha) = -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} \zeta(-2-\alpha).$$

Proof. From [44, Th. 4.11]

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O(N^{-\operatorname{Re} s}) = \zeta_N(s) - \frac{N^{1-s}}{1-s} + O(N^{-\operatorname{Re} s}) \quad (2.21)$$

uniformly for $\operatorname{Re} s > 0$, $|\operatorname{Im} s| < 2\pi N/C$, where C is a given constant greater than 1, and $\zeta_N(s)$ was defined in (2.10). Then from (2.21) and the definition (2.20) of $A_N(s)$ we have for such values of s :

$$A_N(s) = \zeta(s) - \frac{1}{2} \frac{1}{N^s} - O(N^{-\operatorname{Re} s}).$$

This implies that for $\operatorname{Re} s > 0$ we have $\lim_{N \rightarrow \infty} A_N(s) = \zeta(s)$. Then $\operatorname{Re} \alpha < -2$ implies

$$\lim_{N \rightarrow \infty} A_N(-2-\alpha) = \zeta(-2-\alpha)$$

and by the definition of $D_N(\alpha)$, for $\operatorname{Re} \alpha < -2$:

$$\begin{aligned} &\lim_{N \rightarrow \infty} D_N(\alpha) \\ &= \lim_{N \rightarrow \infty} \left[-\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} A_N(-2-\alpha) + \frac{\pi^2}{6} \frac{(2\pi N)^{\alpha+1}}{\alpha+1} \right] \\ &= -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} \zeta(-2-\alpha). \quad \square \end{aligned}$$

Corollary 2.3. *Let $D(\alpha)$ be as in (2.7), then for $\operatorname{Re} \alpha < -2$:*

$$D(\alpha) = -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} \zeta(-2-\alpha). \quad (2.22)$$

Proof. This is immediate from Lemma 2.2 and $D(\alpha) = \lim_{N \rightarrow \infty} D_N(\alpha)$ for $\operatorname{Re} \alpha < -2$. \square

Remark 2.4. 1) We also can express $D(\alpha)$ as a Mellin transform. We first recall that the Mellin transform of a complex-valued continuous function f on the positive real line is defined by

$$M(f)(s) \equiv \int_0^{+\infty} x^{s-1} f(x) dx, \quad s \in \mathbb{C} \quad (2.23)$$

(see, e.g., [30,34]). It exists as an absolutely convergent integral if $x \rightarrow x^{s-1} f(x)$ is in $L^1([0, +\infty))$. Set

$$p_0(y) \equiv \begin{cases} p(y), & \text{for } y \in [1, +\infty), \\ 0, & \text{for } y \in [0, 1), \end{cases}$$

with p as in Section 2 (e.g. (2.2)). Then using the definition of D in (2.7) we see that

$$D(\alpha) = M(p_0)(\alpha + 1),$$

for all $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha < -1$. In this way $D(\alpha)$ appears as the Mellin transform of the function p_0 evaluated at $\alpha + 1$.

2) Let us derive an expression of D as a convergent series containing incomplete gamma functions. We have namely from (2.7) and (2.2), for $\operatorname{Re} \alpha < -1$:

$$\begin{aligned} D(\alpha) &= \int_1^{+\infty} y^\alpha \sum_{n=1}^{+\infty} \frac{\cos ny}{n^2} dy = \sum_{n=1}^{+\infty} \frac{1}{n^2} \int_1^{+\infty} y^\alpha \cos ny dy \\ &= \frac{1}{2} \sum_{n=1}^{+\infty} \frac{i^{-\alpha-1}}{n^{3+\alpha}} [(-1)^{-\alpha-1} \Gamma(\alpha + 1, -in) + \Gamma(\alpha + 1, in)], \end{aligned} \quad (2.24)$$

where we used dominated convergence to interchange sum and integral (since $|\sum_{n=1}^N \frac{\cos ny}{n^2}| \leq \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ for all $N \in \mathbb{N}$) and

$$\int_1^{+\infty} x^\nu e^{ixy} dx = (-iy)^{-\nu-1} \Gamma(\nu + 1, -iy),$$

for $\operatorname{Re} \nu < 0$, cf. [34, 3.4, p. 199],

$$\Gamma(\lambda, z) = \int_z^{+\infty} t^{\lambda-1} e^{-t} dt = z^{\frac{1}{2}\lambda-\frac{1}{2}} e^{-\frac{1}{2}z} W_{\frac{\lambda}{2}-\frac{1}{2}, \frac{1}{2}\lambda}(z)$$

([34, p. 255], $W_{k,\nu}(z)$ being a Whittaker function, [34, p. 254]). All the series in formula (2.24) are absolutely convergent for $\operatorname{Re} \alpha < -1$.

3. The representation of ζ in terms of the associated Clausen function p

Integral representations of ζ are known since the original work [38]. In particular we mention the one given by Riemann

$$\zeta(s) = \Gamma\left(\frac{s}{2}\right)^{-1} \pi^{\frac{s}{2}} \left\{ \int_1^{\infty} [x^{\frac{s}{2}} + x^{\frac{1-s}{2}}] \psi(x) dx - \frac{1}{s(1-s)} \right\},$$

$$\psi(x) \equiv \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

(see [38], [17, p. 16] and [44]). Another representation we would like to mention is

$$\zeta(s) = \frac{\Gamma(-s)}{2\pi i} \int_{\Gamma} \frac{(-x)^s}{e^x - 1} \frac{dx}{x},$$

Γ being a contour which begins at $+\infty$, descends the real axis, circles the singularity at the origin once in the positive direction, and returns up to the positive real axis to $+\infty$, and where $(-x)^s = \exp[s \log(-x)]$ is defined in the usual way for $-x$ not on the negative real axis (see, e.g., [17, p. 137]). Other somewhat similar representations are given by the Riemann Siegel integral formula (see, e.g., [17, p. 166]). Also the remainder in the approximation formula for $\zeta(s)$ has an integral form:

$$\zeta(s) = \zeta_N(s) - \frac{N^{1-s}}{1-s} - s \int_N^{\infty} \{t\} t^{-s-1} dt,$$

where $\{t\}$ denotes the fractional part of t and $\operatorname{Re} s > 0$ (see, e.g., [43, p. 144]). Further integral representations are given, e.g., by [40, pp. 102, 103, Eqs. 41–43, for all $s \in \mathbb{C}$; Eqs. 44–46, for $0 < \operatorname{Re} s < 1$; Eqs. 48, 50, for $\operatorname{Re} s > 1$; Eqs. 49, for $\operatorname{Re} s > 0$]. Another representation for $\operatorname{Re} s > 1$ is given in [39, p. 405] and for $\operatorname{Re} s > 0$ in [23], see also [44, p. 14].

The main aim of this section is to state and prove Theorem 3.1 which gives a new integral representation of ζ for $\operatorname{Re} s > 0$. This representation is in terms of the dilogarithm function, via the function D (introduced in (2.7) and discussed in Section 2). We shall also deduce some consequences from our representation.

Theorem 3.1. *For any $\operatorname{Re} s > 0$ we have*

$$\zeta(s) = \frac{2s(1+s)}{(2\pi)^{1-s}} \left\{ \frac{\pi^2}{6} \frac{1}{1+s} - \frac{\pi}{2} \frac{1}{s} - \frac{1}{4} \frac{1}{1-s} - D(-2-s) \right\} \quad (3.1)$$

where D is as in Section 2 (formula (2.7)).

Proof. From Corollary 2.3 we have, for $\operatorname{Re} \alpha < -2$:

$$D(\alpha) = -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+1)(\alpha+2)} \zeta(-\alpha-2).$$

Setting $s = -\alpha - 2$ we have for $\operatorname{Re} s > 0$:

$$\begin{aligned} D(-2-s) &= -\frac{\pi^2}{6} \frac{1}{-s-1} + \frac{\pi}{2} \frac{1}{-s} - \frac{1}{4} \frac{1}{-s+1} - \frac{(2\pi)^{1-s}}{2(-s-1)(-s)} \zeta(s) \\ &= \frac{\pi^2}{6} \frac{1}{s+1} - \frac{\pi}{2} \frac{1}{s} - \frac{1}{4} \frac{1}{1-s} - \frac{(2\pi)^{1-s}}{2(s+1)s} \zeta(s), \end{aligned}$$

from which the theorem follows by noticing that $(2\pi)^{1-s} \neq 0$. \square

Let us derive some consequences from Theorem 3.1 (Remarks 3.2, 3.3 and 3.4).

Remark 3.2. The representation in Theorem 3.1 gives a new proof of the fact that $\zeta(s)$ is a meromorphic function in $\operatorname{Re} s > 0$, with a simple pole at $s = 0$. In fact $s \rightarrow D(-2-s)$ is holomorphic in $\operatorname{Re} s > 0$, since it is equal to $M(p_0)(-1-s)$, which is holomorphic in $\operatorname{Re} s > 0$ (and even in $\operatorname{Re} s > -1$), being the Mellin transform at $-1-s$ of the function p_0 and $x \rightarrow x^{-2-u} p_0(x)$ is integrable on \mathbb{R}_+ (cf., e.g., [44, ch. I, 1.29, pp. 46–47]). The other functions entering Theorem 3.1 are meromorphic, with a simple pole at $s = 1$.

Remark 3.3. If s_0 is a zero of ζ , with $\operatorname{Re} s_0 > 0$, then

$$D(-2-s_0) = \frac{\pi^2}{6} \frac{1}{1+s_0} - \frac{\pi}{2} \frac{1}{s_0} - \frac{1}{4} \frac{1}{1-s_0} \quad (3.2)$$

(since $s_0 \neq 0$ and also $1+s_0 \neq 0$, because $\operatorname{Re}(1+s_0) = 1 + \operatorname{Re} s_0 > 1$). In particular if $s_0 = u_0 + iv_0$, $u_0 > 0$, $v_0 \in \mathbb{R}$, then

$$\begin{aligned} D(-2-s_0) &= \frac{\pi^2}{6} \frac{1}{1+u_0+iv_0} - \frac{\pi}{2} \frac{1}{u_0+iv_0} - \frac{1}{4} \frac{1}{1-u_0-iv_0} \\ &= \frac{\pi^2}{6} \frac{1+u_0-iv_0}{(1+u_0)^2+v_0^2} - \frac{\pi}{2} \frac{u_0-iv_0}{u_0^2+v_0^2} - \frac{1}{4} \frac{1-u_0-iv_0}{(1-u_0)^2+v_0^2}. \end{aligned} \quad (3.3)$$

In particular

$$\operatorname{Re} D(-2-s_0) = \frac{\pi^2}{6} \frac{1+u_0}{(1+u_0)^2+v_0^2} - \frac{\pi}{2} \frac{u_0}{u_0^2+v_0^2} - \frac{1}{4} \frac{1-u_0}{(1-u_0)^2+v_0^2}, \quad (3.4)$$

$$\operatorname{Im} D(-2-s_0) = v_0 \left\{ -\frac{\pi^2}{6} \frac{1}{(1+u_0)^2+v_0^2} + \frac{\pi}{2} \frac{1}{u_0^2+v_0^2} - \frac{1}{4} \frac{1}{(1-u_0)^2+v_0^2} \right\}. \quad (3.5)$$

For the special case $u_0 = \frac{1}{2}$ we get then

$$D\left(-\frac{5}{2}-iv_0\right) = \frac{\pi^2}{6} \frac{1}{\frac{3}{2}+iv_0} - \frac{\pi}{2} \frac{1}{\frac{1}{2}+iv_0} - \frac{1}{4} \frac{1}{\frac{1}{2}-iv_0}. \quad (3.6)$$

From this it follows:

$$\operatorname{Re} D\left(-\frac{5}{2}-iv_0\right)=\frac{1}{2}\left\{\frac{\pi^2}{6}\frac{3}{\left(\frac{3}{2}\right)^2+v_0^2}-\frac{\pi}{2}\frac{1}{\left(\frac{1}{2}\right)^2+v_0^2}-\frac{1}{4}\frac{1}{\left(\frac{1}{2}\right)^2+v_0^2}\right\}, \quad (3.7)$$

$$\operatorname{Im} D\left(-\frac{5}{2}-iv_0\right)=v_0\left\{-\frac{\pi^2}{6}\frac{1}{\left(\frac{3}{2}\right)^2+v_0^2}+\frac{\pi}{2}\frac{1}{\left(\frac{1}{2}\right)^2+v_0^2}-\frac{1}{4}\frac{1}{\left(\frac{1}{2}\right)^2+v_0^2}\right\}. \quad (3.8)$$

These formulae can be used for the numerical verification whether s_0 is a zero of the ζ on the critical line (this obviously relies on an efficient evaluation of the integral

$$D\left(-\frac{5}{2}-iv_0\right)=\int_1^{+\infty} y^{-\frac{5}{2}-iv_0} p(y) dy,$$

with p defined in Section 2). Possibly also results on the distribution of zeros of ζ on the critical line can be obtained in this way. Numerical work in this direction is planned.

One could ask the question whether it could be possible to exclude that there exists a sequence $s_0(n) = u_0 + iv_0(n)$ with $1 > u_0 > 0$, $u_0 \neq \frac{1}{2}$, $v_0(n) \rightarrow +\infty$ satisfying (3.3) for all n , this implying that there are at most finitely many zeros of ζ on $\operatorname{Re} s = u_0$. For $u_0 = \frac{1}{2}$ one knows, however, that there is such a sequence, by Hardy's proof of the existence of infinitely many zeros of ζ on the critical line, see, e.g., [44, ch. 11]. A comparison of the first and second order in an asymptotic expansion in powers of $\frac{1}{v}$ for $v \rightarrow \infty$ of the two members of (3.3) gives a negative answer to this question, in the sense that it does not permit to distinguish between the behavior at $u_0 = \frac{1}{2}$ and at $u_0 \neq \frac{1}{2}$ (up to these orders in $\frac{1}{v}$).

Remark 3.4. Let us briefly indicate how one can deduce in a simple way a zero-free region for ζ close to the real line, using the integral representation of ζ given in Theorem 3.1.

From Theorem 3.1 if $s_0 = u_0 + iv_0$ satisfies $u_0 > 0$ and $\zeta(s_0) = 0$ then

$$\frac{2s_0(1+s_0)}{(2\pi)^{1-s_0}}\left\{\frac{\pi^2}{6}\frac{1}{1+s_0}-\frac{\pi}{2}\frac{1}{s_0}-\frac{1}{4}\frac{1}{1-s_0}-D(-2-s_0)\right\}=0. \quad (3.9)$$

Since $s_0 \neq 0$, $1+s_0 \neq 0$ this implies, if $v_0 \neq 0$, dividing by $2v_0s_0(1+s_0)/(2\pi)^{1-s_0}$:

$$\frac{1}{v_0}\left[\frac{\pi^2}{6}\frac{1}{1+s_0}-\frac{\pi}{2}\frac{1}{s_0}-\frac{1}{4}\frac{1}{1-s_0}\right]=\frac{1}{v_0}D(-2-s_0). \quad (3.10)$$

But

$$\begin{aligned} \operatorname{Im}\left(\frac{D(-2-s)}{v}\right) &= -\int_1^{+\infty} x^{-2-u} \frac{\sin(v \log x)}{v} p(x) dx \\ &= -\int_1^{+\infty} x^{-2-u} \frac{\sin(v \log x)}{v \log x} (\log x) p(x) dx. \end{aligned} \quad (3.11)$$

For any $u > 0$ define

$$c(u) \equiv - \int_1^{+\infty} x^{-2-u} (\log x) p(x) dx, \quad u > 0. \quad (3.12)$$

Then we have

$$\begin{aligned} \left| \operatorname{Im} \frac{D(-2-s)}{v} - c(u) \right| &= \left| \int_1^{+\infty} y^{-2-u} \left[\frac{\sin(v \log y)}{v \log y} - 1 \right] (\log y) p(y) dy \right| \\ &\leq \int_1^{+\infty} y^{-2-u} \left| \frac{\sin(v \log y) - v \log y}{v \log y} \right| (\log y) |p(y)| dy. \end{aligned} \quad (3.13)$$

But from Taylor's formula

$$\left| \frac{\sin(v \log y) - v \log y}{v \log y} \right| \leq \frac{v^3(y)(\log y)^3}{3! v \log y} \leq \frac{v^2 (\log y)^2}{3!} \quad (3.14)$$

for all $v > 0$ for some $v^*(y) \in [0, v]$, $y \in [1, +\infty)$. On the other hand the left hand side of (3.14) is also bounded by 2. Introducing these bounds into (3.13) we get

$$\left| \operatorname{Im} \frac{D(-2-s)}{v} - c(u) \right| \leq \frac{v^2}{3!} \int_1^{+\infty} y^{-2-u} (\log y)^3 |p(y)| dy \leq b(u, v), \quad (3.15)$$

with

$$b(u, v) \equiv \frac{\pi^2}{3} \min \left\{ \frac{v^2}{3!} \frac{1}{2} \int_1^{+\infty} y^{-2-u} (\log y)^3 dy, \int_1^{+\infty} y^{-2-u} (\log y) dy \right\}$$

and where we used $|p(y)| \leq \pi^2/6$, $y \in [1, +\infty)$, $u > 0$.

Assume that $s_0 = u_0 + iv_0$ satisfies $u_0 > 0$ and $\zeta(s_0) = 0$ and set

$$B(u_0, v_0) \equiv \frac{1}{v_0} \left[\frac{\pi^2}{6} \frac{1}{1+s_0} - \frac{\pi}{2} \frac{1}{s_0} - \frac{1}{4} \frac{1}{1-s_0} \right].$$

By Eq. (3.10) one has that $B(u_0, v_0) = \frac{D(-2-s_0)}{v_0}$, then the inequality (3.15) implies that

$$c(u_0) - b(u_0, v_0) \leq \operatorname{Im} B(u_0, v_0) \leq c(u_0) + b(u_0, v_0). \quad (3.16)$$

But

$$\operatorname{Im} B(u_0, v_0) = -\frac{\pi^2}{6} \frac{1}{(1+u_0)^2 + v_0^2} + \frac{\pi}{2} \frac{1}{u_0^2 + v_0^2} - \frac{1}{4} \frac{1}{(1-u_0)^2 + v_0^2}, \quad (3.17)$$

for all $0 < u_0 < 1$.

The inequality (3.16) is certainly not satisfied, i.e., $s_0 = u_0 + iv_0$ is not a zero of the function $\zeta(s)$, if

$$\operatorname{Im} B(u_0, v_0) > c(u_0) + b(u_0, v_0) \quad (3.18)$$

or

$$\operatorname{Im} B(u_0, v_0) < c(u_0) - b(u_0, v_0). \quad (3.19)$$

But $\max_{u_0 \in (0,1)} b(u_0, v_0) \leq \frac{\pi^2}{3} \min\{\frac{v_0^2}{2}, 1\}$, where we used that for any $\alpha < -1$:

$$\int_1^{+\infty} y^\alpha (\log y)^3 dy = \frac{6}{(\alpha+1)^4}, \quad \int_1^{+\infty} y^\alpha (\log y) dy = \frac{1}{(\alpha+1)^2}$$

(as seen by integrations by parts). From (3.17) it follows that for $u_0 \in (0, 1)$ (3.18) is certainly satisfied for values (u_0, v_0) such that

$$\begin{aligned} \frac{\pi^2}{2} \frac{1}{u_0^2 + v_0^2} &> \max_{u_0 \in (0,1)} c(u_0) + \frac{\pi^2}{3} \min\left\{\frac{v_0^2}{2}, 1\right\} \\ &+ \frac{\pi^2}{6} \frac{1}{(1+u_0)^2 + v_0^2} + \frac{1}{4} \frac{1}{(1-u_0)^2 + v_0^2}. \end{aligned} \quad (3.20)$$

To check the inequality (3.20) we need a lower bound for $\max_{u_0 \in (0,1)} c(u_0)$, this can be obtained numerically as follows: first we rewrite $c(u_0)$ as

$$\begin{aligned} c(u_0) &= - \int_1^{2\pi N} x^{-2-u_0} (\log x) p(x) dx - \int_{2\pi N}^{+\infty} x^{-2-u_0} (\log x) p(x) dx \\ &\equiv c_N(u_0) + r_N(u_0), \end{aligned} \quad (3.21)$$

where $N \in \mathbb{N}$. Then we notice that $p(x) \geq 0$ for $x \in [2\pi n, 2\pi n + \pi \frac{3-\sqrt{3}}{3}] \cup [2\pi n + \pi \frac{3+\sqrt{3}}{3}, 2\pi n + 2\pi]$, for any $n = 0, 1, 2, \dots$, and $p(x) \leq 0$ for $x \in [2\pi n + \pi \frac{3-\sqrt{3}}{3}, 2\pi n + \pi \frac{3+\sqrt{3}}{3}]$, for any $n = 0, 1, 2, \dots$, and we rewrite $c_N(u_0)$ as the sum of its positive and negative part, resp. $c_{N,+}(u_0)$ and $c_{N,-}(u_0)$:

$$c_N(u_0) = c_{N,+}(u_0) - c_{N,-}(u_0). \quad (3.22)$$

Since $c_{N,+}(u_0)$ and $c_{N,-}(u_0)$ are decreasing functions of u_0 , for any $N \in \mathbb{N}$ and $0 < u_0 < 1$, the equality (3.22) gives

$$c_{N,+}(1) - c_{N,-}(0) \leq c_N(u_0) \leq c_{N,+}(0) - c_{N,-}(1). \quad (3.23)$$

To get a lower and an upper bound for $r_N(u_0)$ for $0 < u_0 < 1$ we notice that for any $x > 0$, we have $-\frac{\pi^2}{12} \leq p(x) \leq \frac{\pi^2}{6}$. Then we get

$$m_N \equiv -\frac{\pi^2}{6} \int_{2\pi N}^{+\infty} x^{-2} (\log x) dx \leq r_N(u_0) \leq \frac{\pi^2}{12} \int_{2\pi N}^{+\infty} x^{-2} (\log x) dx \equiv M_N. \quad (3.24)$$

From the formula (3.21) and from the bounds (3.23) and (3.24) it follows that

$$c_{N,+}(1) - c_{N,-}(0) + m_N \leq c(u_0) \leq c_{N,+}(0) - c_{N,-}(1) + M_N.$$

The terms in the upper and lower bound can be easily computed numerically. E.g. for $N = 100$ we obtain

$$-0.12 \lesssim c(u_0) \lesssim 0.36. \quad (3.25)$$

Now using the numerical estimate given in (3.25) we see that (3.20) holds for values of $u_0 > 0$ near 0 and values of $v_0 \leq 1.1$ (e.g. for $u_0 = 0.1$, $v_0 = 1.1$ we have ~ 4.05 at the left hand side and ~ 3.15 at the right hand side of the latter inequality). Let us remark that the bound (3.19) does not seem to yield better results. As it stands the zero-free region is smaller than the one obtained in [1] (by another method, see also this reference for other zero-free regions). Obviously even within the limit of what can be reached from Theorem 3.1 our considerations are not optimal in many directions. E.g. one could restrict u_0 to smaller intervals, such as $u_0 \in (\frac{1}{2}, 1)$, to get slightly better bounds for $c(u_0)$. But in order to obtain really stronger results one would require better numerical or analytical techniques to handle the function D .

4. Other representations of ζ in terms of periodic functions

Set

$$\tilde{p}(x) \equiv p(x) + \frac{\pi^2}{12}, \quad x \in \mathbb{R}. \quad (4.1)$$

Since $\min p = p(\pi) = -\pi^2/12$ the function \tilde{p} satisfies $\tilde{p}(x) \geq 0$ for all $x \in \mathbb{R}$. From the definition (2.1) of p , $\tilde{p}(x)$ can be written as

$$\tilde{p}(x) = \frac{1}{4}(x - (2n+1)\pi)^2, \quad x \in [2\pi n, 2\pi(n+1)), \quad n \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}.$$

For $\operatorname{Re} \alpha < -1$, set

$$\tilde{D}(\alpha) \equiv \int_1^{+\infty} y^\alpha \tilde{p}(y) dy = D(\alpha) - \frac{\pi^2}{12} \frac{1}{\alpha+1}. \quad (4.2)$$

Remark 4.1. From Corollary 2.3 and Theorem 3.1, and from the definition (4.2) of \tilde{D} , it follows that for any $\operatorname{Re} \alpha < -2$ and $\operatorname{Re} s > 0$ we have

$$\tilde{D}(\alpha) = -\frac{\pi^2}{4} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{\alpha+3}}{2(\alpha+2)(\alpha+1)} \zeta(-2-\alpha),$$

hence the representation of ζ given in Theorem 3.1 can also be written in the form:

$$\zeta(s) = \frac{2s(1+s)}{(2\pi)^{1-s}} \left\{ \frac{\pi^2}{4} \frac{1}{1+s} - \frac{\pi}{2} \frac{1}{s} - \frac{1}{4} \frac{1}{1-s} - \tilde{D}(-2-s) \right\}, \quad \operatorname{Re} s > 0. \quad (4.3)$$

We define the function

$$q(x) \equiv \frac{1}{2}(x - (2n+1)\pi), \quad x \in [2\pi n, 2\pi(n+1)), \quad n \in \mathbb{N}_0. \quad (4.4)$$

We observe that $q(x) \equiv \frac{d}{dx} \tilde{p}(x) = \frac{d}{dx} p(x)$ for any $x \in (2\pi n, 2\pi(n+1))$ and $n \in \mathbb{N}_0$.
For any $\operatorname{Re} \alpha < -1$ set

$$E(\alpha) \equiv \int_1^{+\infty} y^\alpha q(y) dy. \quad (4.5)$$

Proposition 4.2. Let $E(\alpha)$ be as in (4.5), then for $\operatorname{Re} \alpha < -1$:

$$E(\alpha) = \frac{\pi}{2} \frac{1}{\alpha+1} - \frac{1}{2} \frac{1}{\alpha+2} + \frac{(2\pi)^{2+\alpha}}{2(\alpha+1)} \zeta(-1-\alpha). \quad (4.6)$$

Proof. The proof follows closely what was done to obtain the formula (2.22) in Corollary 2.3. For any integer $N > 1$, set

$$\begin{aligned} E_N(\alpha) &\equiv \int_1^{2\pi N} y^\alpha q(y) dy = \int_1^{2\pi} y^\alpha q(y) dy + \sum_{k=1}^{N-1} \int_{2\pi k}^{2\pi(k+1)} y^\alpha q(y) dy \\ &= \frac{1}{2} \int_1^{2\pi} y^\alpha (y - \pi) dy + \sum_{k=1}^{N-1} \frac{1}{2} \int_{2\pi k}^{2\pi(k+1)} y^\alpha (y - (2k+1)\pi) dy \\ &= \frac{1}{2} \frac{1}{\alpha+2} [(2\pi)^{\alpha+2} - 1] - \frac{\pi}{2} \frac{1}{\alpha+1} [(2\pi)^{\alpha+1} - 1] \\ &\quad + \sum_{k=1}^{N-1} \left\{ \frac{1}{2} \frac{1}{\alpha+2} [(2\pi(k+1))^{\alpha+2} - (2\pi k)^{\alpha+2}] \right. \\ &\quad \left. - \frac{\pi}{2} \frac{(2k+1)}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \right\} \\ &= \frac{1}{2} \frac{1}{\alpha+2} [(2\pi N)^{\alpha+2} - 1] - \frac{\pi}{2} \frac{1}{\alpha+1} [(2\pi)^{\alpha+1} - 1] \\ &\quad - \sum_{k=1}^{N-1} \frac{\pi}{2} \frac{(2k+1)}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}]. \end{aligned} \quad (4.7)$$

Since

$$\begin{aligned} &\sum_{k=1}^{N-1} \frac{\pi}{2} \frac{(2k+1)}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \\ &= \frac{\pi}{2} \frac{(2\pi)^{\alpha+1}}{\alpha+1} \sum_{k=1}^{N-1} (2k+1) [(k+1)^{\alpha+1} - k^{\alpha+1}] \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{2} \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[2 \sum_{k=1}^{N-1} (k+1)^{\alpha+2} - \sum_{k=1}^{N-1} (k+1)^{\alpha+1} - 2 \sum_{k=1}^{N-1} k^{\alpha+2} - \sum_{k=1}^{N-1} k^{\alpha+1} \right] \\
&= \frac{\pi}{2} \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[2[N^{\alpha+2} - 1] - \sum_{k=2}^N k^{\alpha+1} - \sum_{k=1}^N k^{\alpha+1} + N^{\alpha+1} \right] \\
&= \frac{\pi}{2} \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[2N^{\alpha+2} + N^{\alpha+1} - 1 - 2 \sum_{k=1}^N k^{\alpha+1} \right].
\end{aligned}$$

We use the formula (2.21), see, e.g., [44], as we did in the proof of Lemma 2.2, and obtain

$$\begin{aligned}
&\sum_{k=1}^{N-1} \frac{\pi}{2} \frac{(2k+1)}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \\
&= \frac{\pi}{2} \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[2N^{\alpha+2} + N^{\alpha+1} - 1 - 2\zeta(-\alpha-1) - 2 \frac{N^{\alpha+2}}{\alpha+2} + O(N^{\operatorname{Re}(\alpha+1)}) \right] \\
&= \frac{\pi}{2} \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[2 \frac{\alpha+1}{\alpha+2} N^{\alpha+2} - 1 - 2\zeta(-\alpha-1) + O(N^{\operatorname{Re}(\alpha+1)}) \right],
\end{aligned}$$

which holds true uniformly for $\operatorname{Re}(\alpha) < -1$, $|\operatorname{Im} \alpha| < 2\pi N/C$, where C is a given constant greater than 1.

We use the latter equality in Eq. (4.7) for $E_N(\alpha)$ and obtain

$$E_N(\alpha) = -\frac{1}{2} \frac{1}{\alpha+2} + \frac{\pi}{2} \frac{1}{\alpha+1} + \frac{1}{2} \frac{(2\pi)^{\alpha+2}}{\alpha+1} \zeta(-\alpha-1) + O(N^{\operatorname{Re}(\alpha+1)}).$$

The statement follows from

$$E(\alpha) = \lim_{N \rightarrow \infty} E_N(\alpha) = -\frac{1}{2} \frac{1}{\alpha+2} + \frac{\pi}{2} \frac{1}{\alpha+1} + \frac{1}{2} \frac{(2\pi)^{\alpha+2}}{\alpha+1} \zeta(-\alpha-1)$$

for all $\alpha < -1$. \square

Corollary 4.3. *The following integral representation for ζ in terms of the function E defined by (4.4), (4.5) holds, for all $\operatorname{Re} s > 0$:*

$$\zeta(s) = \frac{2s}{(2\pi)^{1-s}} \left[-\frac{\pi}{2} \frac{1}{s} - \frac{1}{2} \frac{1}{1-s} - E(-1-s) \right]. \quad (4.8)$$

Proof. This is an immediate consequence of Proposition 4.2. \square

Remark 4.4. One could use (4.8) to deduce explicit zero-free regions for $\zeta(s)$, similarly as in Remark 3.4. Moreover from (4.8) and $|q(y)| \leq \pi/2$ we get

$$|E(-1-u-iv)| = \left| \int_1^{+\infty} y^{-1-u-iv} q(y) dy \right| \leq \int_1^{+\infty} y^{-1-u} |q(y)| dy \leq \frac{\pi}{2} \frac{1}{u}.$$

From this and (4.8) we deduce the following simple explicit bound, for all $\operatorname{Re} s > 0$:

$$\begin{aligned} |\zeta(u + iv)| &\leq \frac{1}{(2\pi)^{1-u}} \left[\pi + \frac{|u + iv|}{|1 - u - iv|} + 2|u + iv| |E(-1 - u - iv)| \right] \\ &\leq \frac{1}{(2\pi)^{1-u}} \left[\pi + \frac{u + |v|}{|v|} + \pi \frac{(u + |v|)}{u} \right] \\ &= \frac{1}{(2\pi)^{1-u}} \left[\frac{\pi}{u} |v| + 2\pi + 1 + \frac{u}{|v|} \right]. \end{aligned} \quad (4.9)$$

For $u \geq \frac{1}{2}$ our bound is of the type given in [44, 2.12.2] (namely $\zeta(s) = O(|v|)$, for $u \geq \frac{1}{2}$). To improve our bound one should exploit the oscillatory nature of the integrand, which would require a separate analysis.

For other bounds on $|\zeta(s)|$ see, e.g., [17, p. 184], [25, pp. 116–118, 125], [24, pp. 38, 104, 106, 113, 196], [44, p. 113].

Remark 4.5. By comparison of Eqs. (4.3) and (4.8) we have

$$\tilde{D}(-2 - s) = \frac{1}{1 + s} \left[\frac{1}{4} (1 - \pi)^2 + E(-1 - s) \right]$$

for any $\operatorname{Re} s > 0$. The same result can be obtained directly from the definition (4.2) of $\tilde{D}(\alpha)$, by splitting the integral into the domains $[2\pi k, 2\pi(k + 1))$ and integrating by parts.

Whereas the representation of $\zeta(s)$ given by Theorem 3.1 involves $D(-2 - s)$, which is built with the function $p(x)$ which is quadratic in the fundamental domain $[0, 2\pi)$, the one given by Remark 4.4 involves $E(-1 - s)$, which is built with the function $q(x)$ which is linear in the fundamental domain $[0, 2\pi)$ (being the derivative of p). The next considerations will involve a function $f(x)$ which is constant in the fundamental domain.

Set

$$f(x) \equiv (-1)^n, \quad x \in [2\pi n, 2\pi(n + 1)), \quad n \in \mathbb{N}_0$$

and for all $\operatorname{Re} \alpha < -1$:

$$F(\alpha) \equiv \int_1^{+\infty} y^\alpha f(y) dy. \quad (4.10)$$

Then we have

Proposition 4.6. Let $F(\alpha)$ be as in (4.10), then for $\operatorname{Re} \alpha < -1$:

$$F(\alpha) = -\frac{1}{\alpha + 1} + 2 \frac{(2\pi)^{\alpha+1}}{\alpha + 1} (1 - 2^{2+\alpha}) \zeta(-\alpha - 1). \quad (4.11)$$

Proof. For any integer $N \in \mathbb{N}$, set

$$\begin{aligned} F_N(\alpha) &\equiv \int_1^{2\pi N} y^\alpha f(y) dy = \int_1^{2\pi} y^\alpha dy + \sum_{k=1}^{N-1} (-1)^k \int_{2\pi k}^{2\pi(k+1)} y^\alpha dy \\ &= \frac{1}{\alpha+1} [(2\pi)^{\alpha+1} - 1] + \sum_{k=1}^{N-1} (-1)^k \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}]. \end{aligned} \quad (4.12)$$

We notice that

$$\begin{aligned} &\sum_{k=1}^{N-1} (-1)^k \frac{1}{\alpha+1} [(2\pi(k+1))^{\alpha+1} - (2\pi k)^{\alpha+1}] \\ &= \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[\sum_{k=2}^N (-1)^{k-1} k^{\alpha+1} - \sum_{k=1}^{N-1} (-1)^k k^{\alpha+1} \right] \\ &= \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[-1 + \sum_{k=1}^N (-1)^{k-1} k^{\alpha+1} + \sum_{k=1}^{N-1} (-1)^{k-1} k^{\alpha+1} \right] \\ &= \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[-1 + 2 \sum_{k=1}^N (-1)^{k-1} k^{\alpha+1} - (-1)^{N-1} N^{\alpha+1} \right]. \end{aligned}$$

Using the latter equality in Eq. (4.12) for $F_N(\alpha)$ we obtain

$$\begin{aligned} F_N(\alpha) &= \frac{1}{\alpha+1} [(2\pi)^{\alpha+1} - 1] + \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[-1 + 2 \sum_{k=1}^N (-1)^{k-1} k^{\alpha+1} - (-1)^{N-1} N^{\alpha+1} \right] \\ &= -\frac{1}{\alpha+1} + \frac{(2\pi)^{\alpha+1}}{\alpha+1} \left[2 \sum_{k=1}^N (-1)^{k-1} k^{\alpha+1} - (-1)^{N-1} N^{\alpha+1} \right]. \end{aligned}$$

Since for all $\operatorname{Re} \alpha < -1$ the series on the right hand side of the latter equation converges and $\lim_{N \rightarrow \infty} F_N(\alpha) = F(\alpha)$, then

$$F(\alpha) = \lim_{N \rightarrow \infty} F_N(\alpha) = -\frac{1}{\alpha+1} + 2 \frac{(2\pi)^{\alpha+1}}{\alpha+1} \sum_{k=1}^{\infty} (-1)^{k-1} k^{\alpha+1}$$

for any $\operatorname{Re} \alpha < -1$. The statement follows from

$$\sum_{k=1}^{\infty} (-1)^{k-1} k^{\alpha+1} = (1 - 2^{2+\alpha}) \zeta(-\alpha - 1) \quad (4.13)$$

for any $\operatorname{Re} \alpha < -1$, see [44, Sec. 2.2]. \square

Corollary 4.7. From Proposition 4.6 we get the integral representation for the ζ -function in terms of F :

$$\zeta(s) = \frac{1}{2} \frac{(2\pi)^s}{1 - 2^{1-s}} - \frac{s}{2} \frac{(2\pi)^s}{1 - 2^{1-s}} F(-1 - s), \quad (4.14)$$

for any $\operatorname{Re} s > 0$.

Remark 4.8. This representation yields the simple bound

$$\begin{aligned} |\zeta(u + iv)| &\leq \frac{(2\pi)^u}{2} \frac{1}{\sqrt{1 + 2^{2-2u} - 2^{2-u} \cos(v \log 2)}} [1 + |s| |F(-1 - s)|] \\ &\leq \frac{(2\pi)^u}{2} \frac{1}{\sqrt{1 + 2^{2-2u} - 2^{2-u} \cos(v \log 2)}} \left[1 + \frac{|s|}{u} \right], \end{aligned}$$

where in the latter step we used $|F(-1 - s)| \leq \int_1^{+\infty} y^{-1-u} dy = \frac{1}{u}$. The growth in $|s|$ of the bound is similar to the one obtained in Remark 4.4. To sharpen the bound one should again exploit the oscillations of the integrand in $F(-1 - s)$.

We also remark that using (4.13) we get from Corollary 4.7, for $\operatorname{Re} s > 0$:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \frac{s}{2} (2\pi)^s \left[\frac{1}{s} - F(-1 - s) \right],$$

which is the known representation of the ζ -function as a convergent alternating series, see, e.g., [44, Sect. 2.2].

5. Relations with the Müntz formula

Müntz's formula (see [29] and [44, p. 29]) relates the Mellin transform of a modified theta transform of a test function f with the product of the Mellin transform of the test function and the ζ -function. This formula is presented here in a generalized form, assuming only that f and its Fourier transform $\mathcal{F}(f)$ are in $L^1(\mathbb{R})$, see next Proposition 5.1. Further in this section we present an $s \rightarrow 1 - s$ symmetrized version (under a stronger condition on f and $\mathcal{F}(f)$) of this generalized Müntz formula (Proposition 5.6). We then relate Müntz formula for a certain choice $f = \varphi$ of f to our integral representation for the ζ -function given by Theorem 3.1 (this is the content of Remark 5.11)

Proposition 5.1 (Generalized Müntz formula). Let $f \in L^1(\mathbb{R})$, and assume that $\mathcal{F}(f) \in L^1(\mathbb{R})$, $\mathcal{F}(f)(x) \equiv \int_{\mathbb{R}} e^{2\pi ixy} f(y) dy$ being the Fourier transform of f . Then:

$$1) \quad \Theta_N(f)(x) \equiv \sum_{n=1}^N f(nx)$$

converges absolutely for all $x > 0$ to

$$\Theta(f)(x) \equiv \sum_{n=1}^{\infty} f(nx).$$

2) For any $0 < \operatorname{Re}(s) < 1$ the following statements hold:

a) the Mellin transform of f

$$M(f)(s) \equiv \int_0^{\infty} x^{s-1} f(x) dx$$

exists in the sense of Lebesgue integrals;

b) define $\check{\Theta}(f)$ by

$$\check{\Theta}(f)(x) \equiv \Theta(f)(x) - \frac{1}{x} \int_0^{\infty} f(y) dy, \quad x > 0,$$

then the Mellin transform of $\check{\Theta}(f)$ exists in the sense of Lebesgue integrals and one has

$$M(f)(s)\zeta(s) = M(\check{\Theta}(f))(s).$$

Proof. This is a consequence of a theorem of Müntz [32] (see also [44, pp. 28–29, 2.11]), combined with the absolute convergence of the integrals and almost sure convergence of the series in $\check{\Theta}(f)(x)$ being made clear in the work by Burnol [11, Sect. 3.2, Prop. 3.15], to which we also refer for more details. Note that the assumptions are weaker than in [32] but are covered by the result of [11]. \square

Remark 5.2. Müntz formula has been analyzed, extended and exploited in very interesting recent work by J.-F. Burnol, see, e.g. [7–11], and Báez-Duarte, see, e.g., [2]. It has also been exploited (independently of above work) for the study of zero-free regions of ζ in [1]. In algebraic contexts similar formulae appear in pioneering work by Tate [42] and, more recently, in work by Connes [12] (see also, e.g. [13]) and Meyer [31].

We shall now rewrite Müntz's formula in a way which exploits better the intrinsic basic symmetry with respect to $s \rightarrow 1 - s$ used classically for deriving the functional equation (which one obtains in the particular case where $f(x) = e^{-\pi x^2}$). For this one uses Poisson's summation formula:

Proposition 5.3 (Poisson summation formula). *Let $f \in L^1(\mathbb{R})$ and $\mathcal{F}(f)$ (defined as in Proposition 5.1) be continuous and assume they satisfy*

$$|f(x)| + |\mathcal{F}(f)(x)| \leq \frac{c}{(1 + |x|)^{1+\delta}}$$

for some $c, \delta > 0$, and all $x \in \mathbb{R}$. Then for any $a, b > 0$ such that $ab = 2\pi$ we have

$$\sqrt{a} \sum_{k \in \mathbb{Z}} f(ak) = \sqrt{b} \sum_{k \in \mathbb{Z}} \widehat{f}(bk),$$

with $\widehat{f}(y) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iyx} dx$. Both series are absolutely convergent and converge uniformly to a continuous function of a resp. b .

Proof. $f \in L^1(\mathbb{R})$ implies that $\mathcal{F}(f)$ exist. The rest follows from e.g., [41, Corollary 2.6, (2.8), with $f(x)$ replaced by $\sqrt{a}f(ax)$] see also, e.g., [21,37], [47, Eqs. 13–14, p. 70]. \square

Corollary 5.4. Let f be as in Proposition 5.3. Then

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}} e^{2\pi ixy} f(x) dx = \sqrt{2\pi} \widehat{f}(-2\pi y), \quad y \in \mathbb{R},$$

and for any $a > 0$ we have

$$a \sum_{k \in \mathbb{Z}} f(ak) = \sum_{k \in \mathbb{Z}} \mathcal{F}(f)\left(-\frac{k}{a}\right)$$

(with both series being absolutely convergent to continuous functions of a).

Proof. This is immediate from Proposition 5.3 with $b = 2\pi/a$ and from the fact that

$$\widehat{f}(by) = \frac{1}{\sqrt{2\pi}} \mathcal{F}(f)\left(-\frac{by}{2\pi}\right)$$

by the definitions of \widehat{f} and $\mathcal{F}(f)$. \square

Corollary 5.5. Let f be as in Proposition 5.3 and assume that $y \mapsto f(y)$ is an even function ($y \in \mathbb{R}$). Then for $x > 0$

$$\Theta(f)(x) = \frac{1}{2} \left[\frac{\mathcal{F}(f)(0)}{x} - f(0) \right] + \frac{1}{x} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right).$$

Proof. We first remark that f being even implies $\mathcal{F}(f)$ being even; on the other hand, from Corollary 5.4 we have

$$a \sum_{k \in \mathbb{N}} f(ak) + af(0) + a \sum_{k \in \mathbb{N}} f(-ak) = \sum_{k \in \mathbb{N}} \mathcal{F}(f)\left(-\frac{k}{a}\right) + \mathcal{F}(f)(0) + \sum_{k \in \mathbb{N}} \mathcal{F}(f)\left(\frac{k}{a}\right).$$

Both f and $\mathcal{F}(f)$ are even, therefore we get

$$2a \sum_{k \in \mathbb{N}} f(ak) + af(0) = 2 \sum_{k \in \mathbb{N}} \mathcal{F}(f)\left(\frac{k}{a}\right) + \mathcal{F}(f)(0).$$

From this, replacing a by x , and using the definition of Θ we get

$$2x\Theta(f)(x) + xf(0) = 2\Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) + \mathcal{F}(f)(0), \quad x > 0,$$

which proves Corollary 5.5. \square

Proposition 5.6 (Symmetrized version of the generalized Müntz formula). Let f be even and as in Proposition 5.3. Then for any $0 < \operatorname{Re}(s) < 1$:

$$M(f)(s)\zeta(s) = \frac{1}{2} \left[\frac{\mathcal{F}(f)(0)}{s-1} - \frac{f(0)}{s} \right] + I(f)(s), \quad (5.1)$$

with

$$I(f)(s) \equiv \int_1^\infty x^{s-1} \Theta(f)(x) dx + \int_1^\infty x^{-s} \Theta(\mathcal{F}(f))(x) dx. \quad (5.2)$$

Both integrals on the right hand side of this formula for $I(f)$ exist in Lebesgue's sense.

Proof. From Proposition 5.1 we have for any $0 < a < 1 < b < +\infty$:

$$M(f)(s)\zeta(s) = \lim_{a \downarrow 0} \int_a^1 x^{s-1} \check{\Theta}(f)(x) dx + \lim_{b \uparrow \infty} \int_1^b x^{s-1} \check{\Theta}(f)(x) dx. \quad (5.3)$$

Both limits exist and the integrals are in Lebesgue's sense. From the definition of $\check{\Theta}$ (in Proposition 5.1) and Corollary 5.5, then

$$\begin{aligned} \int_a^1 x^{s-1} \check{\Theta}(f)(x) dx &= \frac{1}{2} \int_a^1 x^{s-1} \left[\frac{\mathcal{F}(f)(0)}{x} - f(0) \right] dx \\ &\quad + \int_a^1 x^{s-1} x^{-1} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) dx - \int_a^1 x^{s-1} \frac{\mathcal{F}(f)(0)}{2x} dx \\ &= -\frac{f(0)}{2} \int_a^1 x^{s-1} dx + \int_a^1 x^{s-2} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) dx \\ &= -\frac{f(0)}{2s} [1 - a^s] + \int_a^1 x^{s-2} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) dx. \end{aligned} \quad (5.4)$$

In the latter expression the first term converges for $a \downarrow 0$ (for all $0 < \operatorname{Re}(s) < 1$) to $-f(0)/(2s)$. Since the limit of the left hand side of (5.4) for $a \downarrow 0$ exists, also absolutely, by Proposition 5.1,

$$\lim_{a \downarrow 0} \int_a^1 x^{s-2} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) dx$$

must also exist (for $0 < \operatorname{Re}(s) < 1$). Thus, under our assumption on s :

$$\lim_{a \downarrow 0} \int_a^1 x^{s-1} \check{\Theta}(f)(x) dx = -\frac{1}{2} f(0) \frac{1}{s} + \lim_{a \downarrow 0} \int_a^1 x^{s-2} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) dx. \quad (5.5)$$

On the other hand, for any $0 < a < 1$ (by the change of variables $x \rightarrow x' = \frac{1}{x}$):

$$\int_a^1 x^{s-2} \Theta(\mathcal{F}(f))\left(\frac{1}{x}\right) dx = \int_{\frac{1}{a}}^1 x'^{2-s} \Theta(\mathcal{F}(f))(x') \left(-\frac{1}{x'^2}\right) dx' = \int_1^{\frac{1}{a}} x^{-s} \Theta(\mathcal{F}(f))(x) dx. \quad (5.6)$$

From (5.5) and (5.6) we get

$$\lim_{a \downarrow 0} \int_a^1 x^{s-1} \check{\Theta}(f)(x) dx = -\frac{1}{2} f(0) \frac{1}{s} + \lim_{a \downarrow 0} \int_1^{\frac{1}{a}} x^{-s} \Theta(\mathcal{F}(f))(x) dx, \quad (5.7)$$

with both limits existing absolutely. By (5.3) and (5.7) we get under our assumptions on s :

$$M(f)(s)\zeta(s) = -\frac{1}{2} f(0) \frac{1}{s} + \lim_{a \downarrow 0} \int_1^{\frac{1}{a}} x^{-s} \Theta(\mathcal{F}(f))(x) dx + \lim_{b \uparrow \infty} \int_1^b x^{s-1} \check{\Theta}(f)(x) dx.$$

We note that from Proposition 5.1 the existence of the limit of the latter integral is assured. But, from the definition of $\check{\Theta}$:

$$\begin{aligned} \int_1^b x^{s-1} \check{\Theta}(f)(x) dx &= \int_1^b x^{s-1} \Theta(f)(x) dx - \int_1^b x^{s-1} \frac{\mathcal{F}(f)(0)}{2x} dx \\ &= \int_1^b x^{s-1} \Theta(f)(x) dx - \frac{\mathcal{F}(f)(0)}{2(s-1)} (b^{s-1} - 1). \end{aligned} \quad (5.8)$$

Since the second term in the latter expression converges for $b \rightarrow +\infty$ to $\frac{\mathcal{F}(f)(0)}{2(s-1)}$, also the first term must converge, as $b \rightarrow +\infty$. Hence we get

$$M(f)(s)\zeta(s) = \frac{1}{2} \left(\frac{\mathcal{F}(f)(0)}{s-1} - \frac{f(0)}{s} \right) + \int_1^\infty x^{-s} \Theta(\mathcal{F}(f))(x) dx + \int_1^\infty x^{s-1} \Theta(f)(x) dx, \quad (5.9)$$

which proves (5.1) (with the integrals converging absolutely). \square

Remark 5.7. We note that in Proposition 5.1 only the values of f on \mathbb{R}_+ are used. Corollary 5.5 (under the stronger assumption on f given in Proposition 5.3) serves to replace the integral over \mathbb{R}_+ on the right hand side of Proposition 5.1 by integrals on $[1, +\infty)$, and here an even extension of a given f on \mathbb{R}_+ to \mathbb{R} is used in order to deduce Proposition 5.6.

Proposition 5.8. Let f be even and as in Proposition 5.3. Then for any $0 < \operatorname{Re} s < 1$:

$$1) \quad I(\mathcal{F}(f))(s) = I(f)(1-s), \quad (5.10)$$

$$2) \quad M(\mathcal{F}(f))(s)\zeta(s) = M(f)(1-s)\zeta(1-s) \quad (5.11)$$

where $I(f)$ was defined in (5.2).

Proof. We first remark that by the assumptions in Proposition 5.3, we have that both f and $\mathcal{F}(f)$ are in $L^1(\mathbb{R})$. Moreover $I(f)(s)$ and $I(\mathcal{F}(f))(s)$ exist in Lebesgue's sense for any $0 < \operatorname{Re} s < 1$. Moreover being f even, then $\mathcal{F}(f)$ is also even. We also have, $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$, $x \in \mathbb{R}$ (as follows easily from the definition of \mathcal{F} and f , $\mathcal{F}(f) \in L^1(\mathbb{R})$, see, e.g. [46, p. 173]). But f being even, we then get $\mathcal{F}(\mathcal{F}(f)) = f$. Then by (5.2) one has

$$\begin{aligned} I(\mathcal{F}(f))(s) &= \int_1^\infty x^{s-1} \Theta(\mathcal{F}(f))(x) dx + \int_1^\infty x^{-s} \Theta(\mathcal{F}(\mathcal{F}(f)))(x) dx \\ &= \int_1^\infty x^{s-1} \Theta(\mathcal{F}(f))(x) dx + \int_1^\infty x^{-s} \Theta(f)(x) dx = I(f)(1-s) \end{aligned}$$

which proves the formula (5.10). Moreover by Proposition 5.6, $M(\mathcal{F}(f))(s)\zeta(s)$ can be written by using Eq. (5.1). Formula (5.11) is a trivial consequence of (5.10) and of the fact that $\mathcal{F}(\mathcal{F}(f))(0) = f(0)$ implies

$$\frac{1}{2} \left(\frac{\mathcal{F}(\mathcal{F}(f))(0)}{s-1} - \frac{\mathcal{F}(f)(0)}{s} \right) = \frac{1}{2} \left(\frac{f(0)}{s-1} - \frac{\mathcal{F}(f)(0)}{s} \right). \quad \square$$

Remark 5.9. From formula (5.11) and from the functional equation $\Gamma(z/2)\pi^{-\frac{z}{2}}\zeta(z) = \Gamma((1-z)/2)\pi^{\frac{1-z}{2}}\zeta(1-z)$, for all $z \in \mathbb{C}$ (see, e.g., [44], [20, 9.535] and [35]) it follows that, for any f satisfying the assumptions of Proposition 5.8, the Mellin transform $M(\mathcal{F}(f))$ of $\mathcal{F}(f)$ exists and is given in terms of the Mellin transform of f by

$$\begin{aligned} M(\mathcal{F}(f))(s) &= \frac{\zeta(1-s)}{\zeta(s)} M(f)(1-s) \\ &= \frac{\Gamma(s/2)\pi^{1-s}}{\Gamma((1-s)/2)} M(f)(1-s), \quad 0 < \operatorname{Re} s < 1, \zeta(s) \neq 0. \end{aligned} \quad (5.12)$$

Let us point out that whereas the right hand side of the first equality is well defined only if $\zeta(s) \neq 0$, the right hand side of the second equality is also well defined for all $0 < \operatorname{Re} s < 1$, so that

$$M(\mathcal{F}(f))(s) = \frac{\Gamma(\frac{s}{2})\pi^{1-s}}{\Gamma(\frac{1-s}{2})} M(f)(1-s), \quad (5.13)$$

for all $0 < \operatorname{Re} s < 1$.

In the following we consider a particular choice φ of the test function f in the Müntz formula. This will permit to obtain explicit formulae for both $\mathcal{F}(f)$ and $M(\mathcal{F}(f))$ and, more importantly, to relate our integral representation for ζ with Müntz formula (in the form of (5.15) below, see also Remark 5.11).

Proposition 5.10. Let $\varphi(x) \equiv 1 - |x|$ for $x \in [-1, +1]$, $\varphi(x) \equiv 0$, $x \in (-\infty, -1) \cup (1, +\infty)$. Then:

$$\begin{aligned} 1) \quad \mathcal{F}(\varphi)(y) &= \int_{\mathbb{R}} e^{2\pi ixy} \varphi(x) dx = \frac{1 - \cos(2\pi y)}{2\pi^2 y^2}, \quad y \in \mathbb{R} \setminus \{0\}, \\ \mathcal{F}(\varphi)(0) &= 1, \quad \text{for } y = 0. \end{aligned}$$

2) φ has all properties of the function f in Proposition 5.8.

3) For all $0 < \operatorname{Re} s$ the Mellin transform of φ is given by

$$M(\varphi)(s) = \frac{1}{s(s+1)}. \quad (5.14)$$

For $0 < \operatorname{Re} s < 1$ one has

$$M(\mathcal{F}(\varphi))(s)\zeta(s) = \frac{1}{(1-s)(2-s)}\zeta(1-s). \quad (5.15)$$

For s such that $\zeta(s) \neq 0$, $0 < \operatorname{Re} s < 1$ we also have

$$M(\mathcal{F}(\varphi))(s) = \frac{\zeta(1-s)}{\zeta(s)(1-s)(2-s)}.$$

4) The Mellin transform of $\mathcal{F}(\varphi)$ is given for all $0 < \operatorname{Re} s < 1$ by

$$M(\mathcal{F}(\varphi))(s) = \frac{\Gamma(s/2)\pi^{1-s}}{\Gamma((1-s)/2)} \frac{1}{(1-s)(2-s)}. \quad (5.16)$$

Proof. 1) This is a simple computation (in fact the formula can be extracted, e.g., from [16, p. 186]), minding that

$$\mathcal{F}(\varphi)(y) = (2\pi)^{\frac{1}{2}} \hat{\varphi}(-2\pi y)$$

and

$$\hat{\varphi}(z) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos z}{z^2}, \quad z \in \mathbb{R}.$$

2) φ is even, continuous and obviously satisfies $|\varphi(x)| \leq \frac{c}{(1+|x|)^{1+\delta}}$, for some $c > 0$, $\delta > 0$. Moreover $\mathcal{F}(\varphi)$ is also continuous, since it behaves as $1 - \frac{(2\pi y)^4}{4!2\pi^2 y^2}$ in a neighborhood of 0 (as seen from Taylor's formula) and is bounded by $\frac{2}{2\pi^2 y^2}$ for large $|y|$, hence satisfies $|\hat{\varphi}(x)| \leq \frac{c}{(1+|x|)^{1+\delta}}$, for some $c, \delta > 0$. Let us remark, in addition, that φ is in fact C^∞ on $\mathbb{R} \setminus (\{0\} \cup \{+1\} \cup \{-1\})$, with right and left derivatives at 0 and ± 1 , with finite jumps from the left to the right. $\mathcal{F}(\varphi)$ belongs to $C^\infty(\mathbb{R})$ and is even (as Fourier transform of the even function φ).

3) By the definitions of M , φ

$$\begin{aligned} M(\varphi)(s) &= \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{+\infty} x^{s-1} \varphi(x) dx = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^1 x^{s-1} (1-x) dx \\ &= \lim_{\varepsilon \downarrow 0} \left[\frac{1}{s} x^s \Big|_{\varepsilon}^1 - \frac{1}{s+1} x^{s+1} \Big|_{\varepsilon}^1 \right] = \lim_{\varepsilon \downarrow 0} \left[\frac{1}{s} - \frac{1}{s} \varepsilon^s - \frac{1}{s+1} + \frac{1}{s+1} \varepsilon^{s+1} \right] \\ &= \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)} \end{aligned}$$

(where we used $\operatorname{Re} s > 0$). Formula (5.15) comes from formula (5.11) in Proposition 5.8 and from (5.14).

4) This is an immediate consequence of (5.13) (a consequence of the functional equation for ζ) and (5.14). \square

Remark 5.11. The computation of the Mellin transform in Proposition 5.10, 4), of the function $\mathcal{F}(\varphi)$ in Proposition 5.10, 1), also might have some interest in itself (this Mellin transform does not seem to be contained, e.g., in [33]).

Let us also point out that (5.15) with (5.16) yield, in turn, the well known functional equation for the Riemann zeta function.

Let us stress furthermore that we can derive (5.15) using our integral representation for ζ given in Theorem 3.1. From the definition of φ given in Proposition 5.10 and by using Eq. (5.1) one has namely

$$M(\mathcal{F}(\varphi))(s)\zeta(s) = \frac{1}{2} \frac{1}{s(s-1)} + \int_1^\infty x^{s-1} \Theta(\mathcal{F}(\varphi))(x) dx, \quad (5.17)$$

where we used the definition (5.2) of $I(\varphi)$, the fact that $\mathcal{F}(\mathcal{F}(\varphi)) = \varphi$, and the fact that $\varphi(x) = 0$ for $x > 1$. From the computation of $\mathcal{F}(\varphi)$ in Proposition 5.10, 1) we have, for $y \in [1, +\infty)$

$$\Theta(\mathcal{F}(\varphi))(y) = \sum_{n=1}^{+\infty} \frac{1 - \cos(2\pi ny)}{2\pi^2 n^2 y^2} = \frac{1}{12} \frac{1}{y^2} - \frac{1}{2\pi^2} \frac{1}{y^2} p(2\pi y),$$

where we used $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$ and the definition (2.2) of p . Then

$$\begin{aligned} \int_1^\infty x^{s-1} \Theta(\mathcal{F}(\varphi))(x) dx &= \frac{1}{12} \int_1^{+\infty} x^{s-3} dx - \frac{1}{2\pi^2} \int_1^{+\infty} x^{s-3} p(2\pi x) dx \\ &= -\frac{1}{12} \frac{1}{s-2} - \frac{1}{\pi (2\pi)^{s-1}} \int_1^{2\pi} y^{s-3} p(y) dy \\ &\quad + \frac{1}{\pi (2\pi)^{s-1}} D(s-3). \end{aligned} \quad (5.18)$$

From Corollary 2.3 one has, on the other hand:

$$\begin{aligned} \frac{1}{\pi (2\pi)^{s-1}} D(s-3) &= \frac{1}{\pi (2\pi)^{s-1}} \left[-\frac{\pi^2}{6} \frac{1}{s-2} + \frac{\pi}{2} \frac{1}{s-1} \right. \\ &\quad \left. - \frac{1}{4} \frac{1}{s} - \frac{(2\pi)^s}{2(s-1)(s-2)} \zeta(1-s) \right]. \end{aligned} \quad (5.19)$$

From the definition of p , on the other hand:

$$\begin{aligned} \frac{1}{\pi (2\pi)^{s-1}} \int_1^{2\pi} y^{s-3} p(y) dy &= \frac{1}{\pi (2\pi)^{s-1}} I^{s-3} \\ &= \frac{1}{\pi (2\pi)^{s-1}} \left[\frac{\pi^2}{6} \frac{1}{s-2} ((2\pi)^{s-2} - 1) \right. \\ &\quad \left. - \frac{\pi}{2} \frac{1}{s-1} ((2\pi)^{s-1} - 1) + \frac{1}{4} \frac{1}{s} ((2\pi)^s - 1) \right] \end{aligned} \quad (5.20)$$

where I^a was defined in (2.12) and computed in (2.13). Using (5.19) and (5.20) in (5.18), and then (5.18) in (5.17), it follows that

$$M(\mathcal{F}(\varphi))(s)\zeta(s) = \frac{1}{(s-1)(s-2)}\zeta(1-s),$$

which is formula (5.15).

Remark 5.12. We also remark that computing in another way $M(\mathcal{F}(\varphi))$ we can get an explicit integral, which might have some interest in itself. In fact using Proposition 5.10, 1), for $f = \varphi$, and the definition (2.23) of M we have

$$\begin{aligned} M(\mathcal{F}(\varphi))(s) &= \int_0^1 x^{s-1} \frac{1 - \cos(2\pi x)}{2\pi^2 x^2} dx + \int_1^\infty x^{s-1} \frac{1 - \cos(2\pi x)}{2\pi^2 x^2} dx \\ &= \int_0^1 x^{s-1} \frac{1 - \cos(2\pi x)}{2\pi^2 x^2} dx - \frac{1}{2\pi^2} \frac{1}{s-2} - \frac{1}{2\pi^2} \int_1^\infty x^{s-3} \cos(2\pi x) dx. \end{aligned} \quad (5.21)$$

Using, e.g., [33, 5.24] (for $a = 2\pi$, $c = 1$, $b = 0$, $z = s - 2$):

$$\int_1^\infty x^{s-3} \cos(2\pi x) dx = \frac{1}{2} (2\pi i)^{-(s-2)} \Gamma(s-2, 2\pi i) + \frac{1}{2} (-2\pi i)^{-(s-2)} \Gamma(s-2, -2\pi i).$$

Inserting this into (5.21) and comparing with (5.16) we get

$$\begin{aligned} M(\mathcal{F}(\varphi))(s) &= \frac{\Gamma(s/2)\pi^{1-s}}{\Gamma((1-s)/2)} \frac{1}{(1-s)(2-s)} = \int_0^1 x^{s-1} \frac{1 - \cos(2\pi x)}{2\pi^2 x^2} dx - \frac{1}{2\pi^2} \frac{1}{s-2} \\ &\quad - \frac{1}{4\pi^2} [(2\pi i)^{-(s-2)} \Gamma(s-2, 2\pi i) + (-2\pi i)^{-(s-2)} \Gamma(s-2, -2\pi i)]. \end{aligned} \quad (5.22)$$

From this we can obtain an expression of the integral on the right hand side (“incomplete Mellin transform” of $\mathcal{F}(\varphi)$) in terms of incomplete gamma functions, namely for $0 < \operatorname{Re} s < 1$:

$$\begin{aligned} \int_0^1 x^{s-1} \frac{1 - \cos(2\pi x)}{2\pi^2 x^2} dx &= \frac{\Gamma(s/2)\pi^{1-s}}{\Gamma((1-s)/2)} \frac{1}{(1-s)(2-s)} + \frac{1}{2\pi^2} \frac{1}{s-2} \\ &\quad - (2\pi i)^{-s} [\Gamma(s-2, 2\pi i) + \Gamma(s-2, -2\pi i)]. \end{aligned} \quad (5.23)$$

Formula (5.23) is used in the following proposition to express a certain series containing factorials in terms of incomplete gamma functions.

Proposition 5.13. *The following summation formula holds for all $0 < \operatorname{Re} s < 1$:*

$$\frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}}{(2k)!} \frac{(-1)^k}{s-2+2k} = \frac{\Gamma(s/2)\pi^{1-s}}{\Gamma((1-s)/2)} \frac{1}{(1-s)(2-s)} + \frac{1}{2\pi^2} \frac{1}{s-2} - (2\pi i)^{-s} [\Gamma(s-2, 2\pi i) + \Gamma(s-2, -2\pi i)]. \quad (5.24)$$

Proof. This follows from (5.23) observing that the left hand side can be expressed in the following way (by inserting the power series expansion of $\cos(2\pi x)$ and using dominated convergence to exchange sum and integration):

$$\int_0^1 x^{s-1} \frac{1 - \cos(2\pi x)}{2\pi^2 x^2} dx = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}}{(2k)!} \frac{(-1)^k}{s-2+2k}. \quad (5.25)$$

Comparison of (5.23) and (5.25) immediately yields the summation formula (which does not seem to appear in the usual tables on series, e.g., [22]). \square

Acknowledgments

This paper is part of a larger research program. Previous collaborations and discussions with Zdzisław Brzeźniak, Christof Cebulla, Benedetta Ferrario, Andrej Madrecki, Danilo Merlini, Lev Pustynnikov, Walter R. Schneider and Bogusław Zegarliński on topics related to the present work helped us in understanding some of the underlying problems and are gratefully acknowledged. S.A. is also grateful to Christoph Berns and Masha Gordina for a careful reading of versions of a previous manuscript, pointing out several necessary improvements, and making many very useful comments for improving it. We are also very grateful to an anonymous referee for very careful reading of the paper, for many suggestions and much constructive criticism, which also greatly helped to improve the presentation of the results.

The warm hospitality and support of the Department of Mathematics, University of Trento and of C.I.R.M., the Scuola Normale Superiore and the Centro de Giorgi in Pisa, are also gratefully acknowledged. One of the authors, C.C., is thankful to the Hausdorff Research Institute for Mathematics and to the JSPS (Japan Society for the Promotion of Science) for financial support. We also thank Dominik Gunkel for his skillful help in type setting the manuscript.

Appendix A. Another derivation of the formula for $D(\alpha)$

We sketch another derivation of the formula for $D(\alpha)$ which enters our integral representation for ζ given in Theorem 3.1. We do not provide all the details but point out some explicit formulae which might have some interest in themselves.

Lemma A.1. *Let $D(\alpha)$ be defined by (2.7), and assume $\operatorname{Re} \alpha < -1$. Then:*

$$D(\alpha) = \frac{\pi^2}{6} \frac{1}{\alpha+1} [(2\pi)^{\alpha+1} - 1] - \frac{\pi}{2} \frac{1}{\alpha+2} [(2\pi)^{\alpha+2} - 1] + \frac{1}{4} \frac{1}{\alpha+3} [(2\pi)^{\alpha+3} - 1] + \tilde{A}(\alpha),$$

with

$$\begin{aligned} \tilde{A}(\alpha) \equiv & \sum_{k=1}^{+\infty} (2\pi k)^\alpha \left\{ \frac{\pi^2}{6} (2\pi) \sum_{l=0}^{+\infty} \binom{\alpha}{l} \frac{1}{l+1} \frac{1}{k^l} - \frac{\pi}{2} (2\pi)^2 \sum_{l=0}^{+\infty} \binom{\alpha}{l} \frac{1}{l+2} \frac{1}{k^l} \right. \\ & \left. + \frac{1}{4} (2\pi)^3 \sum_{l=0}^{+\infty} \binom{\alpha}{l} \frac{1}{l+3} \frac{1}{k^l} \right\}. \end{aligned}$$

Proof. Splitting the integration domain $[1, +\infty)$ in (2.7) into

$$[1, 2\pi) \cup [2\pi, +\infty) = [1, 2\pi) \cup \bigcup_{k=1}^{+\infty} [2\pi k, 2\pi(k+1))$$

we get

$$\int_1^{+\infty} y^\alpha p(y) dy = \int_1^{2\pi} y^\alpha p(y) dy + \tilde{I}^\alpha, \quad \text{with } \tilde{I}^\alpha \equiv \sum_{k=1}^{+\infty} \int_{2\pi k}^{2\pi(k+1)} y^\alpha p(y) dy. \quad (\text{A.1})$$

The series is absolutely convergent, due to the fact that $p(\cdot)$ is uniformly bounded and $\operatorname{Re} \alpha < -1$. The first integral at the right hand side has been computed in (2.13).

Set

$$\tilde{I}^\alpha \equiv \sum_{k=1}^{+\infty} \int_{2\pi k}^{2\pi(k+1)} y^\alpha p(y) dy. \quad (\text{A.2})$$

By the change of variables $y \rightarrow y' = y - 2\pi k$ and using the periodicity of p we get

$$\tilde{I}^\alpha = \sum_{k=1}^{+\infty} \tilde{I}_k^\alpha, \quad (\text{A.3})$$

with

$$\tilde{I}_k^\alpha \equiv \int_0^{2\pi} (y + 2\pi k)^\alpha p(y) dy = (2\pi k)^\alpha \int_0^{2\pi} \left(1 + \frac{y}{2\pi k}\right)^\alpha p(y) dy, \quad k \in \mathbb{N}.$$

Using the binomial series expansion and Lebesgue's dominated convergence, we get

$$\tilde{I}_k^\alpha = (2\pi k)^\alpha \sum_{l=0}^{+\infty} \binom{\alpha}{l} \frac{1}{(2\pi k)^l} \int_0^{2\pi} y^l p(y) dy. \quad (\text{A.4})$$

Performing the integrals, using the definition (2.1) of $p(y)$ in $[0, 2\pi)$, we get easily

$$\begin{aligned} \tilde{I}^\alpha &= \sum_{k=1}^{\infty} (2\pi k)^\alpha \left\{ \frac{\pi^2}{6} \cdot 2\pi \sum_{l=0}^{\infty} \binom{\alpha}{l} \frac{1}{l+1} \frac{1}{k^l} - \frac{\pi}{2} (2\pi)^2 \sum_{l=0}^{\infty} \binom{\alpha}{l} \frac{1}{l+2} \frac{1}{k^l} \right. \\ &\quad \left. + \frac{1}{4} (2\pi)^3 \sum_{l=0}^{\infty} \binom{\alpha}{l} \frac{1}{l+3} \frac{1}{k^l} \right\} = \tilde{A}(\alpha) \end{aligned} \quad (\text{A.5})$$

(where the sums converge absolutely and we used the definition of $\tilde{A}(\alpha)$ in Lemma A.1). Lemma A.1 follows then, introducing (2.13) resp. (A.5) into (A.1). \square

Lemma A.2. Let $\operatorname{Re} \alpha < -1$, then with the notation in Lemma A.1:

$$\begin{aligned} \tilde{A}(\alpha) = & 2^\alpha \frac{\pi^{3+\alpha}}{3} \sum_{l=0}^{+\infty} \frac{1}{l+1} \binom{\alpha}{l} \zeta(l-\alpha) - 2^{1+\alpha} \pi^{3+\alpha} \sum_{l=0}^{+\infty} \frac{1}{l+2} \binom{\alpha}{l} \zeta(l-\alpha) \\ & + 2^{1+\alpha} \pi^{3+\alpha} \sum_{l=0}^{+\infty} \frac{1}{l+3} \binom{\alpha}{l} \zeta(l-\alpha). \end{aligned}$$

Proof. This follows from the expression for $\tilde{A}(\alpha)$ in Lemma A.1 and the power series definition of $\zeta(z)$ with $z = l - \alpha$, observing that: $\operatorname{Re}(l - \alpha) \geq l - \operatorname{Re} \alpha > l + 1 \geq 1$, $l \in \mathbb{N}_0$, so that the series over k converge absolutely to $\zeta(l - \alpha)$. \square

Lemma A.3. For $j = 1, 2, 3$, $\operatorname{Re} \alpha < -2$:

$$\sum_{l=0}^{\infty} \frac{1}{l+j} \binom{\alpha}{l} \zeta(l-\alpha) = \lim_{\eta \downarrow 0} \int_{\eta}^{1-\eta} \zeta(-\alpha, 1+t) t^{j-1} dt = -\frac{1}{\alpha+j} + \tilde{A}_j(\alpha),$$

where $\zeta(\rho, a)$ is the generalized Riemann zeta function, defined for $a \notin -\mathbb{N}_0$, $\operatorname{Re} \rho > 1$ by $\zeta(\rho, a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^\rho}$ and meromorphically continued to all $\rho \in \mathbb{C}$ (see, e.g., [18, vol. 1, 24], [27]), and $\tilde{A}_1(\alpha) \equiv 0$, $\tilde{A}_2(\alpha) \equiv \frac{1}{\alpha+1} \zeta(-\alpha-1)$, $\tilde{A}_3(\alpha) \equiv \frac{1}{\alpha+1} \zeta(-\alpha-1) - \frac{2}{(\alpha+1)(\alpha+2)} \zeta(-\alpha-2)$.

Proof. From [22, 54.12.2, p. 359] or [40, pp. 281, 286]:

$$\sum_{j=0}^{\infty} \frac{(\rho)_j}{j!} \zeta(\rho+j, a) t^j = \zeta(\rho, a-t), \quad (\text{A.6})$$

for $|t| < |a|$, $\rho, \alpha \in \mathbb{C}$, $a-t \notin -\mathbb{N}_0$. Here $(\rho)_j \equiv \rho(\rho+1)\dots(\rho+j-1) = \Gamma(\rho+j)/\Gamma(\rho)$ is Pochhammer's symbol, $j \in \mathbb{N}$, $(\rho)_0 \equiv 1$. From, e.g., [30, p. 4] we have, for any $l \in \mathbb{N}$:

$$\binom{b}{l} = (-1)^l \frac{\Gamma(l-b)}{l! \Gamma(-b)}, \quad b \in \mathbb{C} \setminus \mathbb{N}_0.$$

From this and the definition of $(\rho)_j$ with $\rho = -b$, $j = l$, we have, using $(-b)_l = \frac{\Gamma(-b+l)}{\Gamma(-b)}$:

$$\binom{b}{l} = (-1)^l \frac{\Gamma(l-b)}{l! \Gamma(-b)} = \frac{(-1)^l}{l!} (-b)_l. \quad (\text{A.7})$$

From this we get

$$\binom{b}{l} \zeta(l-b) = \frac{(-1)^l}{l!} (-b)_l \zeta(l-b),$$

hence

$$(-b)_l \zeta(l-b) = \frac{l!}{(-1)^l} \binom{b}{l} \zeta(l-b). \quad (\text{A.8})$$

Using (A.6) for $\rho = -\alpha$, $j = l$, $a = 1$ and $0 < |t| < 1$, we have on the other hand:

$$\sum_{l=0}^{+\infty} \frac{(-\alpha)_l}{l!} \zeta(l-\alpha) t^l = \zeta(-\alpha, 1-t) \quad (\text{A.9})$$

(where we used that $\zeta(l-\alpha, 1) = \sum_{n=0}^{+\infty} \frac{1}{(n+1)^{l-\alpha}} = \sum_{n'=1}^{+\infty} \frac{1}{(n')^{l-\alpha}} = \zeta(l-\alpha)$, where we set $n' = n+1$, minding that $\operatorname{Re}(l-\alpha) > 1$ since $l - \operatorname{Re} \alpha > 1$). From (A.8) (with $b = \alpha$) we have $(-\alpha)_l \zeta(l-\alpha) = \frac{l}{(-1)^l} \binom{\alpha}{l} \zeta(l-\alpha)$ and we get

$$\sum_{l=0}^{+\infty} \frac{(-\alpha)_l}{l!} \zeta(l-\alpha) t^l = \sum_{l=0}^{+\infty} \frac{1}{l!} \frac{l!}{(-1)^l} \binom{\alpha}{l} \zeta(l-\alpha) t^l.$$

Minding that the left hand side is the same as the left hand side of (A.9) we get

$$\sum_{l=0}^{+\infty} \binom{\alpha}{l} \zeta(l-\alpha) (-t)^l = \zeta(-\alpha, 1-t).$$

Replacing t by $-t$ (which also satisfies $0 < |t| < 1$), and multiplying by t^{j-1} , $j = 1, 2, 3$, we get

$$\sum_{l=0}^{+\infty} \binom{\alpha}{l} \zeta(l-\alpha) t^{l+j-1} = \zeta(-\alpha, 1+t) t^{j-1}. \quad (\text{A.10})$$

Integrating with respect to t on $[\eta, 1-\eta]$, $\frac{1}{2} > \eta > 0$ we get

$$\int_{\eta}^{1-\eta} \sum_{l=0}^{+\infty} \binom{\alpha}{l} \zeta(l-\alpha) t^{l+j-1} dt = \int_{\eta}^{1-\eta} \zeta(-\alpha, 1+t) t^{j-1} dt. \quad (\text{A.11})$$

We shall now write $\zeta(l-\alpha) = [\zeta(l-\alpha) - 1 + 1]$ and insert this into the left hand side of (A.11)

$$\int_{\eta}^{1-\eta} \sum_{l=0}^{+\infty} \binom{\alpha}{l} [\zeta(l-\alpha) - 1] t^{l+j-1} dt + \int_{\eta}^{1-\eta} \sum_{l=0}^{+\infty} \binom{\alpha}{l} t^{l+j-1} dt. \quad (\text{A.12})$$

Using that from (A.8) with $b = \alpha$ we have $|(-\alpha)_l \zeta(l-\alpha)| = l! \binom{\alpha}{l} |\zeta(l-\alpha)|$, we get easily for the N -th approximation of the integrand of the first term in (A.12):

$$\begin{aligned} \left| \sum_{l=0}^N \binom{\alpha}{l} (\zeta(l-\alpha) - 1) \right| &\leq \sum_{l=0}^N \frac{1}{l!} |(-\alpha)_l| |\zeta(l-\alpha) - 1| \\ &\leq \sum_{l=0}^{\infty} \frac{1}{l!} \sqrt{(\operatorname{Re} \alpha)^2 + (\operatorname{Im} \alpha)^2} \dots \sqrt{(\operatorname{Re}(\alpha + (l-1)))^2 + (\operatorname{Im} \alpha)^2} \zeta(l - \operatorname{Re} \alpha), \end{aligned} \quad (\text{A.13})$$

where we used the definition of $(-\alpha)_l$ and $|(-\alpha + q)| = \sqrt{(\operatorname{Re} \alpha + q)^2 + (\operatorname{Im} \alpha)^2}$, $q = 0, \dots, l-1$, together with:

$$|\zeta(l - \alpha) - 1| \leq \sum_{n=2}^{\infty} \left| \frac{1}{n^{\operatorname{Re} \alpha + l}} \right| = \sum_{n=2}^{\infty} \frac{1}{n^{\operatorname{Re} \alpha}} \leq [\zeta(l - \operatorname{Re} \alpha) - 1], \quad l \in \mathbb{N}_0$$

(because $|n^{l-\alpha}| = n^{l-\operatorname{Re} \alpha}$, since $|n^{-iv}| = |e^{-iv \log n}| = 1$, $n \in \mathbb{N}$, $v \in \mathbb{R}$ and the series being absolutely convergent, since $l - \operatorname{Re} \alpha > 2$, for all $l \in \mathbb{N}_0$). But

$$\zeta(l - \operatorname{Re} \alpha) - 1 \leq \frac{l + 1 - \operatorname{Re} \alpha}{l - 1 - \operatorname{Re} \alpha} \frac{1}{2^{l - \operatorname{Re} \alpha}}$$

because $\zeta(\sigma) \leq 1 + \frac{\sigma+1}{\sigma-1} \frac{1}{2^\sigma}$, for $\sigma > 1$ (see, e.g., [24, Ex. 2, p. 48]). Hence we get from (A.13) the following bound:

$$\left| \sum_{l=0}^N \binom{\alpha}{l} (\zeta(l - \alpha) - 1) \right| \leq \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sqrt{[(\operatorname{Re} \alpha) + (l-1)]^2 + (\operatorname{Im} \alpha)^2} \right)^l \frac{l + 1 - \operatorname{Re} \alpha}{l - 1 - \operatorname{Re} \alpha} \frac{1}{2^{l - \operatorname{Re} \alpha}}.$$

The series converges absolutely, as seen from Stirling's formulae. The bound (A.13) is then finite, independent of N . Using Lebesgue's dominated convergence theorem it is then not difficult to prove that

$$\sum_{l=0}^{+\infty} \binom{\alpha}{l} \zeta(l - \alpha) \frac{1}{l+j} [(1-\eta)^{l+j} - \eta^{l+j}] = \int_{\eta}^{1-\eta} \zeta(-\alpha, 1+t) t^{j-1} dt, \quad j = 1, 2, 3. \quad (\text{A.14})$$

The summand on the left hand side is bounded absolutely uniformly in η by 2 (since $(1-\eta)^{l+i} \leq 1$ and $\eta^{l+j} \leq 1/2^{l+j} \leq 1$, $l \in \mathbb{N}_0$, $j = 1, 2, 3$). By a discrete version of the Lebesgue dominated convergence theorem we can interchange the limit $\eta \downarrow 0$ with the summation, getting that the limit for $\eta \downarrow 0$ of the left hand side of (A.14) is equal to $\sum_{l=0}^{+\infty} \binom{\alpha}{l} \zeta(l - \alpha) \frac{1}{l+j}$, which is the left hand side in the formula in Lemma A.3.

On the right hand side of (A.14) we have, using the definition of $\zeta(-\alpha, 1+t)$ in Lemma A.3:

$$\int_{\eta}^{1-\eta} \zeta(-\alpha, 1+t) t^{j-1} dt = \int_{\eta}^{1-\eta} \sum_{k=0}^{\infty} (k + (1+t))^{\alpha} t^{j-1} dt. \quad (\text{A.15})$$

It is not difficult to convince ourselves that one can interchange the sum and the integral obtaining

$$\lim_{\eta \downarrow 0} \sum_{k=0}^{+\infty} \int_{\eta}^{1-\eta} (k + (1+t))^{\alpha} dt = -\frac{1}{\alpha + 1}. \quad (\text{A.16})$$

Similarly, for $j = 2$ we have as a result:

$$\begin{aligned} \sum_{k=0}^{+\infty} \int_{\eta}^{1-\eta} (k + (1+t))^{\alpha} t dt &= \sum_{k=0}^{+\infty} \int_{k+1+\eta}^{k+2-\eta} t'^{\alpha} (t' - k - 1) dt' \\ &= \sum_{k=0}^{+\infty} \int_{k+1+\eta}^{k+2-\eta} t'^{\alpha+1} dt' - \sum_{k=0}^{+\infty} (k+1) \int_{k+1+\eta}^{k+2-\eta} t'^{\alpha} dt'. \end{aligned} \quad (\text{A.17})$$

Using the above result for $j = 1$ with α replaced by $\alpha + 1$ we see that the first sum on the right hand side converges for $\eta \downarrow 0$ to $-1/(\alpha + 2)$.

As for the N -th approximation of the second sum on the right hand side of (A.17) we have

$$\begin{aligned} - \sum_{k=0}^N (k+1) \int_{k+1+\eta}^{k+2-\eta} t'^{\alpha} dt' &= - \sum_{k=0}^N \frac{(k+1)}{\alpha+1} [(k+2-\eta)^{\alpha+1} - (k+1+\eta)^{\alpha+1}] \\ &\xrightarrow{N \rightarrow +\infty} - \sum_{k=0}^{+\infty} \frac{(k+1)}{\alpha+1} [(k+2-\eta)^{\alpha+1} - (k+1+\eta)^{\alpha+1}] \\ &\xrightarrow{\eta \downarrow 0} - \sum_{k=0}^{+\infty} \frac{(k+1)}{\alpha+1} [(k+2)^{\alpha+1} - (k+1)^{\alpha+1}] \\ &= \frac{1}{\alpha+1} \sum_{k=1}^{\infty} k^{\alpha+1} = \frac{1}{\alpha+1} \zeta(-\alpha-1), \end{aligned}$$

where we used $(\operatorname{Re} \alpha + 1) < -1$. Hence

$$\begin{aligned} \sum_{k=0}^N \int_{\eta}^{1-\eta} (k + (1+t))^{\alpha} t dt &\xrightarrow{N \rightarrow +\infty} \sum_{k=0}^{+\infty} \int_{\eta}^{1-\eta} (k + (1+t))^{\alpha} t dt \\ &\xrightarrow{\eta \downarrow 0} -\frac{1}{\alpha+2} + \frac{1}{\alpha+1} \zeta(-\alpha-1). \end{aligned} \quad (\text{A.18})$$

For $j = 3$ one first considers

$$\rho_{N,\eta}(\beta) \equiv \sum_{k=0}^N \int_{\eta}^{1-\eta} (k + (1+t))^{\beta} t^2 dt,$$

for $\operatorname{Re} \beta < -3$ (and then one replaces “by analytic continuation” β by α). By careful considerations similar to these we did for $j = 1, 2$, we obtain

$$\lim_{N \rightarrow \infty} \lim_{\eta \downarrow 0} \rho_{N,\eta}(\beta) = -\frac{1}{\beta+3} + \frac{1}{\beta+1} \zeta(-\beta-1) - \frac{2}{(\beta+1)(\beta+2)} \zeta(-\beta-2).$$

By analytic continuation one then obtains

$$\lim_{\eta \downarrow 0} \sum_{k=0}^{\infty} \int_{\eta}^{1-\eta} (k + (1+t))^{\alpha} t^2 dt = -\frac{1}{\alpha+3} + \frac{1}{\alpha+1} \zeta(-\alpha-1) - \frac{2}{(\alpha+1)(\alpha+2)} \zeta(-\alpha-2). \quad (\text{A.19})$$

The completion of the proof is obtained from (A.16), (A.18), (A.19) combined with the fact that the limit of (A.10) for $\eta \downarrow 0$ is the left hand side in the expression in Lemma A.3 and observing that (A.9) holds. \square

Corollary A.4. \tilde{A} as defined in Lemma A.1 satisfies, for $\operatorname{Re} \alpha < -2$:

$$\tilde{A}(\alpha) = 2^{\alpha} \pi^{3+\alpha} \left(-\frac{1}{3} \frac{1}{\alpha+1} + \frac{2}{\alpha+2} - \frac{2}{\alpha+3} \right) - \frac{(2\pi)^{3+\alpha}}{2(\alpha+1)(\alpha+2)} \zeta(-\alpha-2).$$

Proof. This is immediate from Lemma A.2 and Lemma A.3. \square

Corollary A.5. Let $D(\alpha)$ be as in Lemma A.1. Then for $\operatorname{Re} \alpha < -2$:

$$\begin{aligned} D(\alpha) &= -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} + \frac{2^{\alpha+1}}{2 \cdot 3} \frac{1}{\alpha+1} \pi^{3+\alpha} \\ &\quad - \frac{\pi^{3+\alpha}}{2} \frac{2^{\alpha+2}}{\alpha+2} + \frac{1}{2^2} \frac{1}{\alpha+3} 2^{\alpha+3} \pi^{3+\alpha} \tilde{A}(\alpha) \\ &= -\frac{\pi^2}{6} \frac{1}{\alpha+1} + \frac{\pi}{2} \frac{1}{\alpha+2} - \frac{1}{4} \frac{1}{\alpha+3} - \frac{(2\pi)^{3+\alpha}}{2(\alpha+1)(\alpha+2)} \zeta(-\alpha-2). \end{aligned}$$

This coincides with formula (2.22), and hence yields a new proof of Corollary 2.3.

Proof. This is immediate from Lemma A.1 and Corollary A.4, due to a compensation of terms. \square

References

- [1] S. Albeverio, C. Cebulla, Müntz formula and zero free regions for the Riemann zeta function, *Bull. Sci. Math.* 131 (1) (2007) 12–38.
- [2] L. Báez-Duarte, Möbius convolutions and the Riemann hypothesis, *Int. J. Math. Math. Sci.* 22 (2005) 3599–3608 (English).
- [3] S. Beltraminelli, D. Merlini, S. Sekatskii, A hidden symmetry related to the Riemann hypothesis with the primes into the critical strip, *Asian-Eur. J. Math.* 3 (4) (2010) 555–563.
- [4] E. Bombieri, The Riemann hypothesis, in: *The Millennium Prize Problems*, Clay Math. Inst, Cambridge, MA, 2006, pp. 107–124.
- [5] P. Borwein, S. Choi, B. Rooney, A. Weirathmueller (Eds.), *The Riemann Hypothesis. A Resource for the Afficionado and Virtuoso Alike*, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2008, MR 2463715 (2009k:11002).
- [6] J. Brüder, *Introduction to Analytic Number Theory (Einführung in die analytische Zahlentheorie)*, Springer-Verlag, Berlin–Heidelberg, 1995.
- [7] J.-F. Burnol, Some Hilbert spaces of entire functions associated with the Fourier transform and Dirichlet and Riemann L -functions (Sur certains espaces de Hilbert de fonctions entières, liés à la transformation de Fourier et aux fonctions L de Dirichlet et de Riemann), *C. R. Math. Acad. Sci. Paris* 333 (2001) 201–206.
- [8] J.-F. Burnol, On the “Sonine spaces” associated by de Branges to the Fourier transform (Sur les “espaces de Sonine” associés par de Branges à la transformation de Fourier), *C. R. Math. Acad. Sci. Paris* 335 (8) (2002) 689–692 (French).
- [9] J.-F. Burnol, On an analytic estimate in the theory of the Riemann zeta function and a theorem of Báez-Duarte, *Acta Cient. Venezolana* 54 (2003) 210–215.
- [10] J.-F. Burnol, On Fourier and zeta(s), *Forum Math.* 16 (6) (2004) 789–840 (English).

- [11] J.-F. Burnol, Co-Poisson links (Entrelacement de Co-Poisson), *Ann. Inst. Fourier* 57 (2007) 525–602.
- [12] J. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function, *Selecta Math.* (N.S.) 5 (1) (1999) 29–106 (English).
- [13] J. Connes, M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Amer. Math. Soc. Colloq. Publ., vol. 55, American Mathematical Society, Providence, RI, 2008, MR 2371808 (2009b:58015).
- [14] J.B. Conrey, L -functions and random matrices, in: *Mathematics Unlimited–2001 and Beyond*, Springer, Berlin, 2001, pp. 331–352, MR 1852163 (2002g:11134).
- [15] J.B. Conrey, The Riemann hypothesis, *Notices Amer. Math. Soc.* 50 (3) (2003) 341–353 (English).
- [16] W.F. Donoghue Jr., *Distributions and Fourier Transforms*, Pure Appl. Math., vol. 32, Academic Press, New York–London, 1969.
- [17] H.M. Edwards, *Riemann's Zeta Function*, Pure Appl. Math., vol. 58, Academic Press, New York–London, 1974.
- [18] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vols. I, II, McGraw–Hill Book Company, Inc., New York–Toronto–London, 1953, based, in part, on notes left by Harry Bateman, MR 0058756 (15,419i).
- [19] K. Ford, Zero-free regions for the Riemann zeta function, in: *Number Theory for the Millennium, II*, Urbana, IL, 2000, A K Peters, Natick, MA, 2002, pp. 25–56, MR 1956243 (2003k:11136).
- [20] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, seventh edition, Academic Press, 2007.
- [21] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004.
- [22] E.R. Hansen, *A Table of Series and Products*, Prentice Hall, Englewood Cliff, NJ, 1975.
- [23] A. Ivč, The Riemann zeta-function, in: *The Theory of the Riemann Zeta-Function with Applications*, Wiley–Interscience Publication, John Wiley & Sons Inc., New York, 1985, MR 792089 (87d:11062).
- [24] G.J.O. Jameson, *The Prime Number Theorem*, London Math. Soc. Stud. Texts, vol. 53, Cambridge University Press, Cambridge, 2003, MR 1976226 (2004c:11002).
- [25] A.A. Karatsuba, S.M. Voronin, *The Riemann Zeta-Function*, de Gruyter Exp. Math., vol. 5, Walter de Gruyter & Co., Berlin, 1992, translated from Russian by Neal Koblitz.
- [26] M.L. Lapidus, In search of the Riemann zeros, in: *Strings, Fractal Membranes and Noncommutative Spacetimes*, American Mathematical Society, Providence, RI, 2008, MR 2375028 (2009f:11110).
- [27] A. Laurinćikas, R. Garunkštis, *The Lerch Zeta-Function*, Kluwer Academic Publishers, Dordrecht, 2002, MR 1979048 (2004c:11161).
- [28] L. Lewin, *Dilogarithms and Associated Functions*, Macdonald, London, 1958.
- [29] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland Publishing Co., New York–Amsterdam, 1981.
- [30] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, third enlarged edition, Grundlehren Math. Wiss., Band 52, Springer-Verlag New York, Inc., New York, 1966, MR 0232968 (38 #1291).
- [31] R. Meyer, On a representation of the idele class group related to primes and zeros of L -functions, *Duke Math. J.* 127 (3) (2005) 519–595, MR 2132868 (2006e:11128).
- [32] C.H. Müntz, Beziehungen der Riemannschen ζ -Funktion zu willkürlichen reellen Funktionen, *Mat. Tidsskrift B* 1922 (1922) 39–47 (German).
- [33] F. Oberhettinger, *Tables of Mellin Transform*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.
- [34] F. Oberhettinger, *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer-Verlag, Berlin, 1990.
- [35] A. Ossicini, An alternative form of the functional equation for Riemann's zeta function, *Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia* 56 (2008/2009) 95–111.
- [36] S.J. Patterson, *An Introduction to the Theory of the Riemann Zeta-Function*, Cambridge Stud. Adv. Math., vol. 14, Cambridge University Press, Cambridge, 1988, MR 933558 (89d:11072).
- [37] M.A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, Grad. Stud. Math., vol. 102, American Mathematical Society, Providence, RI, 2009, reprint of the 2002 original.
- [38] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsbericht der Berliner Akad.* (1859) 671–680.
- [39] Z. Rudnick, P. Sarnak, Zeros of principal L -functions and random matrix theory, in: *A Celebration of John F. Nash Jr.*, *Duke Math. J.* 81 (1–2) (1996) 269–322.
- [40] H.M. Srivastava, J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer, Dordrecht, 2001.
- [41] E. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Math. Ser., vol. 32, Princeton University Press, Princeton, NJ, 1971.
- [42] J.T. Tate, Fourier analysis in number fields, and Hecke's zeta-functions, in: *Algebraic Number Theory*, Proc. Instructional Conf., Brighton, 1965, Thompson, Washington, DC, 1967, pp. 305–347, MR 0217026 (36 #121).
- [43] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Stud. Adv. Math., vol. 46, Cambridge University Press, Cambridge, 1995, translated from the second French edition by C.B. Thomas.
- [44] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, second edition, The Clarendon Press, Oxford University Press, New York, 1986, edited and with a preface by D.R. Heath-Brown, MR 882550 (88c:11049).
- [45] M. van Frankenhuysen, Arithmetic progressions of zeros of the Riemann zeta function, *J. Number Theory* 115 (2) (2005) 360–370, MR 2180508 (2006g:11178).
- [46] H.J. Weaver, *Theory of Discrete and Continuous Fourier Analysis*, Wiley–Interscience Publication, John Wiley & Sons Inc., New York, 1989, MR 974113 (90c:42002).
- [47] A. Zygmund, *Trigonometric Series I*, Cambridge University Press, London, 1968, MR 0236587 (38 #4882).